Lecture 3: Environment-based computations

Functions as values. Tail recursion. Structural recursion

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Recap from Lecture 2

What is the λ-calculus?

The smallest language for FP. It consists of

1. A **language** to write expressions, also known as terms.

$$t ::= x \mid \lambda x.t_1 \mid t_1\ t_2$$

   where x is a variable and

   - $\lambda x.t$ is an abstraction with intended reading "the function which, for input x computes the value of t."
     - λx is the binder of the abstraction
     - t is the body (or scope) of the abstraction
   - $t_1\ t_2$ is an application: t_1 is applied to argument t_2.

2. Transformation rules

   α-conversion: $\lambda x.t \rightarrow_\alpha \lambda y.[y/x]t$
   if $[y/x]t$ is a capture-free substitution.
   β-reduction: $(\lambda x.t_1)\ t_2 \rightarrow_\beta [t_2/x]t_1$
   if $[t_2/x]t_1$ is a capture-free substitution.

Racket and the λ-calculus

The λ-calculus is the core language of Racket $\Rightarrow$ Racket recognizes the expressions of the λ-calculus, but we should write them in a slightly different way:

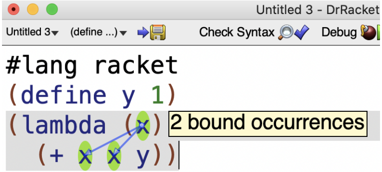
`(lambda (`x`) `t`)` instead of $\lambda x.t$
`(`t_1 t_2`)` instead of $t_1\ t_2$

Remarks

1. For efficiency reasons, Racket has built-in values for many useful datatypes including many predefined functions.
2. The editor of Racket allows us to view the referenced-based representation of λ-terms
   - If we hover the mouse over a binder, the editor highlights the occurrences bound to it (see next slide).
   - If we hover the mouse over a variable occurrence, we see a reference to its corresponding binder (see next slide).

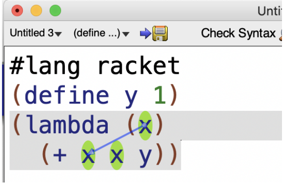
Racket and the λ-calculus

Referenced-based representations of expressions (snapshot)



```
#lang racket
(define y 1)
(lambda (x)
   (+ x x y))
```

2 bound occurrences



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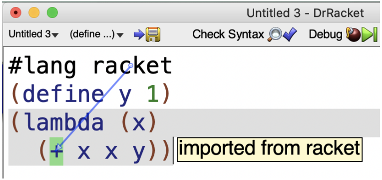


```
#lang racket
(define y 1)
(lambda (x)
   (+ x x y))
```

imported from racket

Transformation rules

The purpose of α-conversion

α-conversion allows us to do harmless renamings of parameters of functions.

Example

Suppose y is a global variable with a given value.

- $\lambda x.y$ is the function which, for every input x, returns the value of y.

  $$\lambda x.y \rightarrow_{\alpha} \lambda z.[z/x]y = \lambda z.y$$

  is harmless because $\lambda x.y$ and $\lambda z.y$ describe the same function. But we are not allowed to perform the variable-capture substitution

  $$\lambda x.y \rightarrow \lambda y.[y/x]y = \lambda y.y$$

  because $\lambda x.y$ and $\lambda y.y$ describe different functions.

Transformation rules

The purpose of β-reduction

β-reduction simulates the first-step of evaluating a function call: We replace in the body of the function the formal parameters with the input arguments.

Example (Evaluation in Racket)

`(define y 7)`

`>` $\underline{\texttt{((lambda (x) (+ x y)) 5)}}$

$\rightarrow_\beta$ `[7/y][5/x](+ x y)`

$=$ $\underline{\texttt{(+ 5 7)}}$

$\rightarrow$ 12

`> ((lambda (x) (lambda (y) (+ x y))) 6)`

$\rightarrow_\beta$ `[6/x](lambda (y) (+ x y))`

$=$ `(lambda (y) (+ 6 y))`

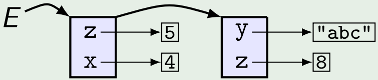
Remark: + and y have free occurrences ⇒ to use them, we need to know where to find their values.

Environment-based computations

Environment = data structure which stores the values of variables with free occurrences.

- Environment = a list of frames.
- Every frame is a table of values for some variables.

Example (Environment *E* with two frames)



- The first frame is the top frame.
- **Variable lookup**: $E(var)$ is the value of *var* found in the first frame, from top to bottom (or left to right) which contains a value for *var*:

  $E(\mathtt{x}) = 4$, $E(\mathtt{y}) = \texttt{"abc"}$, $E(\mathtt{z}) = 5$
  $E(\mathtt{t})$ is not defined.
  The binding $\mathtt{z} \mapsto 8$ is shadowed by the binding $\mathtt{z} \mapsto 5$ in the top frame.

Environment-based computations

Preliminary remarks

All evaluations are performed w.r.t. a **global environment** which stores the values of variables with free occurrences in expressions.

The global environment is initialized with bindings for predefined variables when we start the system

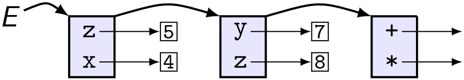
- Built-in functions names are predefined variables with functions as values

The value of an expression *expr* in an environment E is computed in two steps:

1. All variables x in *expr* are replaced with $E(x)$
2. The new expression is evaluated using the rules of evaluation.

Evaluation of expressions

Example



The value of `(+ x (* y z))` in *E* is computed as follows:

$$\texttt{(+ }\underline{\texttt{x}}\texttt{ (* }\underline{\texttt{y}}\texttt{ }\underline{\texttt{z}}\texttt{))}\rightarrow\texttt{(+ 4 }\underline{\texttt{(* 7 5)}}\texttt{)}\rightarrow\underline{\texttt{(+ 4 35)}}\rightarrow 39$$

Remark

From now on we will always assume implicitly that the environment has a frame with bindings for all built-in operations and constants.

Environment-based computations

The interpretation of definitions

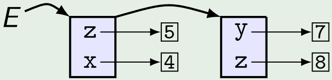
When the interpreter reads a definition

`(define` *var* *expr*`)`

in an environment E, it does the following:

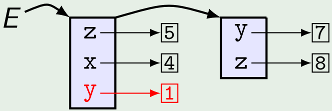
1. It computes the value v of *expr* in E
2. It adds the binding $var \mapsto v$ to the top frame of E.

Example



```
> (+ x (* y z))
39
```

The definition `(define y 1)` changes E to be



```
> (+ x (* y z))
9
```

The new binding $y \mapsto 1$ shadows the binding $y \mapsto 8$.

The interpretation of definitions

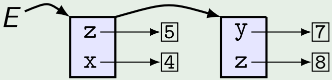
A word of warning

Bindings can shadow each other, but they can not be overwritten

⇒ `(define` *var* *expr*`)`
is prohibited in an environment *E* which has a binding of *var* in the first frame.

Example

We can not redefine `x` and `z` in environment



but we can define `y`.

Blocks and their evaluation

Block = sequence of definitions and expressions, which ends with an expression.

- `(local [ ]` *comp*$_1$ `...` *comp*$_n$ *expr*`)`

  is a special form for the block made of the sequence of components *comp*$_1$, . . . , *comp*$_n$ followed by *expr*.

The evaluation of such a block in an environment E proceeds as follows:

1. E is extended with a temporary top frame, initially empty.
2. The all components of the block are interpreted one by one:
   - the block definitions add bindings to the (initially empty) top frame
   - *expr* is evaluated and its value is returned as value of the block
3. E is restored by discarding its temporary top frame.

The evaluation of blocks

Example

Remark

(`println` *expr*)

prints the value of *expr* on a new line, and returns the value `#<void>`.

We will use `println` to illustrate how block-structured evaluation works

Example

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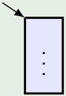

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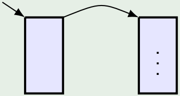

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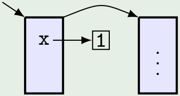

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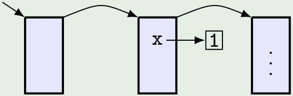

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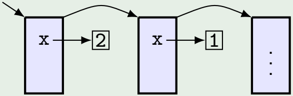

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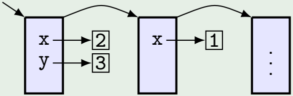

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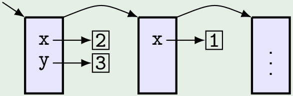
```



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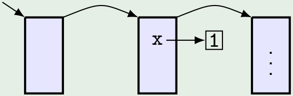

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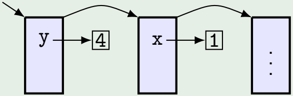

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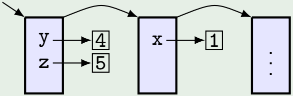

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  10
```

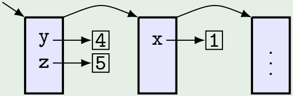

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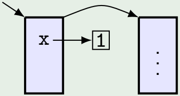

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The evaluation of blocks

Example

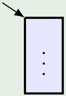
Remark

`(println` *expr*`)`

prints the value of *expr* on a new line, and returns the value `#<void>`.

We will use `println` to illustrate how block-structured evaluation works

Example



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Other special forms with blocks

1. The conditional form

   ```
   (cond [test_1 block_1]
         ...
         [test_n block_n])
   ```

   where $test_1, \ldots, test_n$ are boolean expressions. The evaluation returns the value of the first block $block_i$ for which $test_i$ is true. If all tests are false, the evaluation returns value `#<void>`

2. Abstractions, which are used to define functions

   ```
   (lambda (x_1 ... x_n) block)
   ```

3. `let` and `let*`:

   ```
   (let ([var_1 expr_1]                    (let* ([var_1 expr_1]
         ...                                      ...
         [var_n expr_n])                          [var_n expr_n])
     block)                                  block)
   ```

The boolean operators and and or

`and` and `or` are special forms: **they are not functions!**

1. (and t_1 ... t_n)
   evaluates expressions $t_1, \ldots, t_n$ from left to right.
   - if it finds t_i with value `#f`, it returns `#f`
   - otherwise, it returns the value of t_n.
2. (or t_1 ... t_n)
   evaluates expressions $t_1, \ldots, t_n$ from left to right.
   - if it finds t_i whose value is not `#f`, it returns the value of t_i.
   - otherwise, it returns `#f`.

REMARK: In Racket, all non-`#f` values are true. This is similar to language C, where anything non-zero is interpreted as true.

```
> (and 1 (lambda (x) x) #f)
#f
> (and)
#t
> (and 1 "abc" 'abc)
'abc
```

```
> (or #f 'abc "abc")
'abc
> (or)
#f
```

The special forms `if` and `cond`

(`if` $\mathit{test}\ \ \mathit{expr}_1\ \ \mathit{expr}_2$)

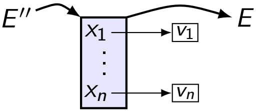
is equivalent with

(`cond` [$\mathit{test}\ \ \mathit{expr}_1$]
        [`#t` expr_2])

- `cond` is more general than `if`, also because its branches can be blocks.
- The branches of `if` must be expressions.

User-defined functions

- The value of `(lambda (`x_1 `...` x_n`)` *block*`)` in an environment E is the pair $\langle$`(lambda (`x_1 `...` x_n`)` *block*`)`,$E\rangle$
  - Such a value is called lexical closure or function closure or closure: it is a pair made of (1) the textual definition of the function and (2) the environment where f was created.
- If f has value $\langle$`(lambda (`x_1 `...` x_n`)` *block*`)`,$E\rangle$ then the value of `(`f t_1 `...` t_n`)` in E' is computed as follows:
  - compute the values $v_1, \ldots, v_n$ of $t_1, \ldots, t_n$ in E'
  - create the temporary environment

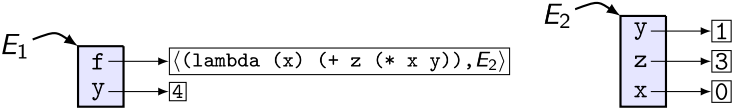


    and compute v =the value of *block* in E''
  - return v as the value of `(f` t_1 `...` t_n`)` in E'.

The evaluation of function calls

Illustrated example

Consider the environments E_1 and E_2 where



What is the value of `(f y)` in E_1?

The evaluation of function calls

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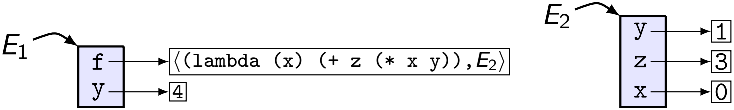
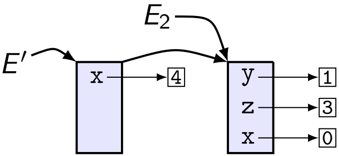


What is the value of `(f y)` in E_1?

$$\texttt{(f }\underline{\texttt{y}}\texttt{) in } E_1 \rightarrow \texttt{(f 4) in } E_1 \rightarrow \texttt{(+ z (* x y)) in } E'$$

where



$>$ $\texttt{(+ }\underline{\texttt{z}}\texttt{ (* }\underline{\texttt{x}}\texttt{ }\underline{\texttt{y}}\texttt{))}$ in E'

$\rightarrow$ $\texttt{(+ 3 (* 4 1))}$ in E'

7

The evaluation of function calls

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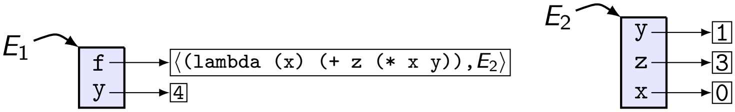
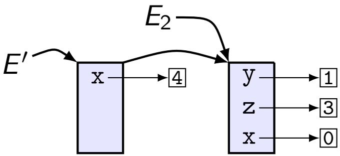


What is the value of (f y) in E_1?

$(\texttt{f}\ \underline{\texttt{y}})\ \texttt{in}\ E_1 \rightarrow (\texttt{f 4})\ \texttt{in}\ E_1 \rightarrow \texttt{(+ z (* x y))}\ \texttt{in}\ E'$

where



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$\rightarrow \texttt{(+ 3 (* 4 1))}\ \text{in}\ E'$

7

⇒ the value of (f y) in E_1 is 12.

Recursive computations

Remarks about recursion

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    - A function is recursive when it calls itself directly or indirectly.
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Why learn recursion?

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- Divide-and-conquer paradigm

Many computations and data structures are naturally recursive

Recursive function definitions

General structure

- A simple base case (or base cases): a terminating scenario that does not use recursion to produce an answer.
- One or more recursive cases that reduce the computation, directly or indirectly, to simpler computations of the same kind.
  - To ensure termination of the computation, the reduction process should eventually lead to base case computations.

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Classic recursive functions:

1. Factorial function
2. Fibonacci function
3. Ackermann function
4. Euclid's Greatest Common Divisor (GCD) function

How to write a recursive definition?

1. Try to break a problem into subparts, at least one of which is similar to the original problem.
    - There may be many ways to do so. For example, if $m, n \in \mathbb{N}$ and $m > n > 0$ then $gcd(m, n) = gcd(m - n, n)$, or $gcd(m, n) = gcd(n, m \mod n)$
2. Make sure that recursion will operate correctly:
    - there should be at least one base case and one recursive case (it's OK to have more)
    - The test for the base case must be performed before the recursive calls.
    - The problem must be broken down such that a base case is always reached in a finite number of recursive calls.
    - The recursive call must not skip over the base case.
    - The non-recursive portions of the subprogram must operate correctly.

Analysis of recursive computations

Case study: computation of the factorial

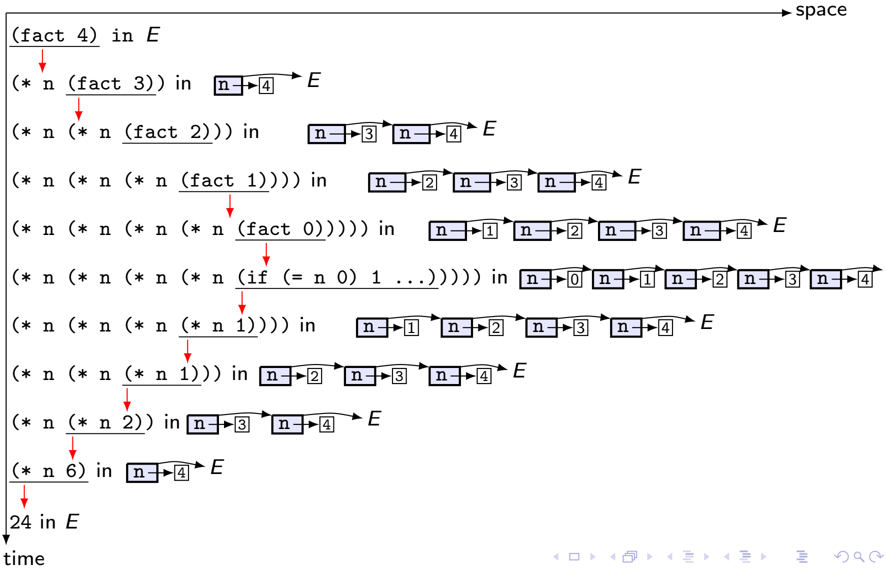
```
(define (fact n)
   (if (= n 0)
       1
       (* n (fact (- n 1)))))
```

Q1: What is the space and time complexity of computing `(fact n)` when $n \in \mathbb{N}$?

The factorial function

Time and space complexity of computation

```
(define (fact n) (if (= n 0) 1 (* n (fact (- n 1)))))
```



(fact 4) in *E*

(* n (fact 3)) in

(* n (* n (fact 2))) in

(* n (* n (* n (fact 1)))) in

(* n (* n (* n (* n (fact 0))))) in

(* n (* n (* n (* n (if (= n 0) 1 ...))))) in

(* n (* n (* n (* n 1)))) in

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(* n 6) in

24 in *E*

Analysis of recursive computations

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A1: The computation of `(fact` n`)` has

time complexity $2 \cdot (n+1) = O(n)$

space complexity $O(n)$: the maximum number of frames added to E is $n+1$

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A2: Main idea: Add an extra argument to accumulate and propagate the result computed so far.

```
(define (fact n) (fact-acc n 1))
(define (fact-acc n a)
   (if (= n 0) a (fact-acc (- n 1) (* a n))))
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```
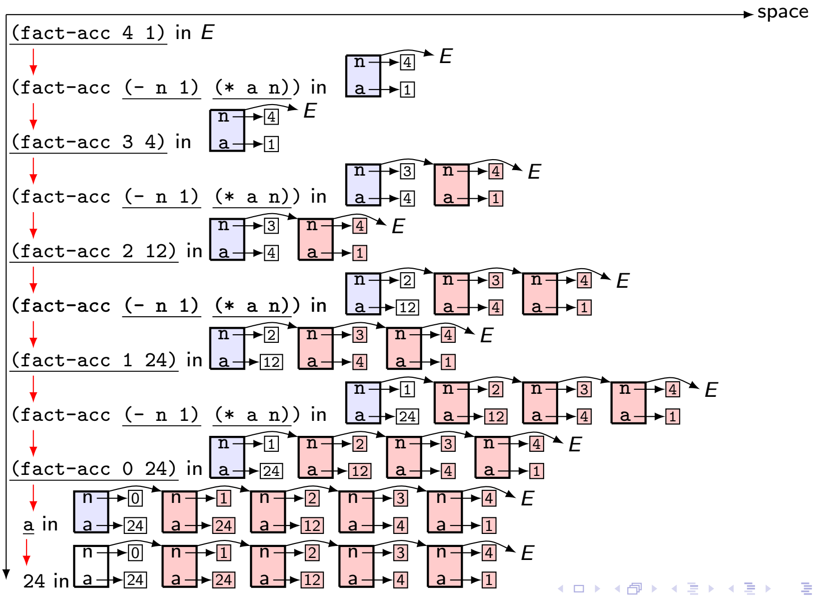

- (`fact-acc` n a) computes $n! \cdot a$, therefore (`fact-acc` n `1`) computes $n!$

The factorial function

Towards a space-efficient implementation



`(fact-acc 4 1)` in *E*

`(fact-acc (- n 1) (* a n))` in

`(fact-acc 3 4)` in

`(fact-acc (- n 1) (* a n))` in

`(fact-acc 2 12)` in

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`(fact-acc 1 24)` in

`(fact-acc (- n 1) (* a n))` in

`(fact-acc 0 24)` in

`a` in

24 in

A space-efficient implementation

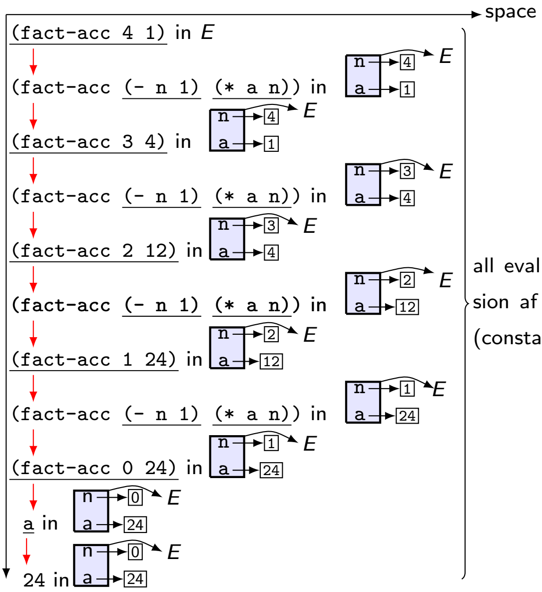
```
(define (fact n)
        (fact-acc n 1))
(define (fact-acc n a)
  (if (= n 0)
      1
      (fact-acc (- n 1) (* a n))))
```

The red-colored frames contain useless information for the computation of the result ⇒ they can be discarded (garbage-collected) by a clever compiler

⇒ the space complexity of computing (`fact-acc` n 1) becomes constant, $O(1)$ (see next slide).

A space-efficient implementation

Example: computation of `(fact-acc 4 1)`



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all evaluations happen in an extension af *E* with at most one frame (constant space of memory)

Tail-call optimization

The space-efficient computation of `(fact-acc 4 1)` is an example of

> tail-call optimization = compiler optimization technique which garbage-collects frames and bindings that become inaccessible.

Remarks

- A function is tail recursive if it is defined such that the recursive call is the last thing executed by the function.
- Tail-call optimization is always applicable to calls of tail recursive functions

  ⇒ **for efficient computation of repetitive computations, try to implement them with tail recursive functions**.

  (+) all iterative computations can be implemented with tail recursion.

  (-) Some recursive computations can not be implemented with tail recursion.

  (+) We will learn techniques to translate some recursive definitions into tail recursive definitions.

Simulating `for` loops by tail recursion

Imperative version

```
for(i = 1; i ≤n;i++)
     body
```

Tail recursive version

```
(f n)
```

where

```
(define (f n [i 1])
   (cond [(<= i n)
      body
      (f n (+ i 1))]))
```

REMARK: In Racket, functions can have optional arguments:

(define (f x_1 ... x_n [y_1 t_1] ... [y_m t_m]) *body*)

declares function f with mandatory arguments $x_1, \ldots, x_n$ and optional arguments $y_1, \ldots, y_m$:

- f can be called with argument values for $y_1, \ldots, y_m$.
- It no inputs are provided for $y_1, \ldots, y_m$, they take the values of $t_1, \ldots, t_m$.

Simulating `for` loops by tail recursion

Examples

1. Print the numbers from 1 to `n`

   ```
   (define (printn n [i 1])
       (cond [(<= i n) (println i) (printn n (+ i 1))]))
   ```

2. Compute the sum of numbers from 1 to `n` by translating

   ```
   for(i = 0; s = 0; i ≤n;i++)   s:=s+i;
   return s;
   ```

   into tail recursive code:

   ```
   ; (sumto n i s) computes the value of s+i+(i+1)...+n
   (define (sumto n [i 0] [s 0])
     (cond [(<= i n) (sumto n (+ i 1) (+ s i))]
           [#t s]))
   ```

REMARK: Every variable modified by a `for` loop becomes a parameter in the tail recursive function definition.

Linear recursive functions are tail recursive

A linear recursive function f of degree k is defined by k initial values $f_1, \ldots, f_k$ as follows:

$$f(n) := \begin{cases} f_n & \text{if } 1 \leq n \leq k \\ c_1 \cdot f(n-1) + c_2 \cdot f(n-2) + \ldots c_k \cdot f(n-k) & \text{if } n > k \end{cases}$$

where $c_1, \ldots, c_k$ are some numeric constants.

Example

The sequence of Fibonacci numbers is defined by linear recursion: $\mathit{fib}(1) = \mathit{fib}(2) = 1, \quad \mathit{fib}(n) = \mathit{fib}(n-1) + \mathit{fib}(n-2)$ if $n > 2$.

General form of a tail recursive definition for f:

```
(define ($f$ n [i 1] [$a_1$ $f_1$] ... [$a_k$ $f_k$])
  (cond [(= i n) $a_1$]
        [#t ($f$ n (+ i 1) $a_2$ ... $a_k$ (+ (* $c_1$ $a_k$)
                                            ...
                                            (* $c_k$ $a_1$)))]))
```

Primitive recursive functions are tail recursive

A primitive recursive function is of the form $h : \mathbb{N} \times A_1 \times \ldots \times A_k \times B \to B$ where

1. $h(0, x_1, \ldots, x_k) := f(x_1, \ldots, x_k)$ (base case)
2. $h(n+1, x_1, \ldots, x_k) := g(n, h(n, x_1, \ldots, x_k), x_1, \ldots, x_k)$ for all $n \in \mathbb{N}$. (recursive case)

Remark

h has a tail recursive definition:

```
(define (h n x_1 ... x_k [i 0]
                         [prev (f x_1 ... x_k)])
  (cond [(= i n) prev]
        [#t (h n x_1 ... x_k (+ i 1)
                             (g i prev x_1 ... x_k))]))
```

Recursive functions with invariant parameters (1)

The parameter `n` in the recursive function definition

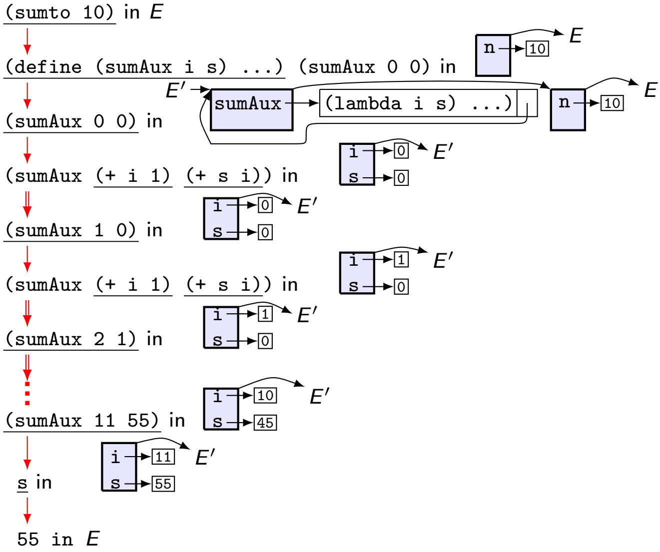
```
(define (sumto n [i 0] [s 0])
  (cond [(<= i n) (sumto n (+ i 1) (+ s i))]
        [#t s]))
```

is invariant, and passing it as argument to recursive calls is time- and space- consuming.
We can avoid passing invariant arguments to recursive calls. For example:

```
(define (sumto n)
  (define sumAux i s)
    (cond [(<= i n) (sumAux (+ i 1) (+ s i))]
          [#t s])
  (sumAux 0 0))
```

Recursive functions with invariant parameters (2)

Suppose *E* is an environment with `sumto` defined in it.



`(sumto 10)` in *E*

`(define (sumAux i s) ...) (sumAux 0 0)` in

`(sumAux 0 0)` in

`(sumAux (+ i 1) (+ s i))` in

`(sumAux 1 0)` in

`(sumAux (+ i 1) (+ s i))` in

`(sumAux 2 1)` in

`(sumAux 11 55)` in

`s` in

`55 in` *E*

Possible pitfalls with recursion

Is recursive computation fast?

- **Yes**: some tail-recursive functions are remarkably efficient
- **No**: We can easily write elegant, but spectacularly inefficient recursive programs, e.g.

```
(define (fib n)
   (if (or (= n 0) (= n 1))
       1
       (+ (fib (- n 1)) (fib (-n 2)))))
```

Recursion can take a long time if it needs to repeatedly recompute intermediate results

General principle: Whenever possible, use tail recursion to make your functions efficient.

Conclusion

Environment-based computation is a standard technique to keep track of the meaning of names in a program.

- Environment = list of frames; every frame is a table that maps distinct names to values.
- Definitions add bindings to the top (=first) frame of the environment
- Evaluation of blocks extends the environment with a temporary top frame, to store the bindings of local definitions. The top frame and its bindings are garbage collected when block evaluation ends.
- In FP, all recursive computations are performed by recursion.
  - Every recursive step extends environment with a new frame ⇒ deep recursive calls produce stack overflow
  - Tail recursion = compiler optimization technique which garbage-collects frames and bindings that become inaccessible