Programming with streams

November, 2018

The purpose of this labwork is to check if you master programming with streams:

- Define certain streams using stream generators implemented with nullary functions
- Define operations on streams

To solve this lab work, you can use the auxiliary functions and streams implemented in `LW5-6.rkt` which can be downloaded from the website of this lecture:

`all-ones`	stream whose elements are all `1`
`(gen-nats n)`	creates the stream of natural numbers starting from `n`
`naturals`	the stream of all natural numbers starting from `0`
`(s-add s1 s2)`	creates the stream which is the componentwise addition of streams `s1` and `s2`
`(s-take n s)`	list of first `n` elements of stream `s`
`(s-filter p s)`	the stream obtained from stream `s` by removing the elements which do not fulfil predicate `p`
`(s-map f s)`	the stream obtained from stream `s` by applying function `f` to all its elements
`fib`	stream of all Fibonacci numbers

L1. Define the operation `(s-drop` n s`)` that returns the stream obtained from stream s by dropping it's first n elements.

L2. Define the operation `(s-between` m n s`)` that returns the list of elements between position m and n (inclusive) in stream s. We assume that the first element of s is at position 1.

<u>Test</u>: The elements of the stream of natural numbers between positions 6 and 8:

```
> (s-between 6 8 nats)
'(5 6 7)
```

L3. Define the operation `(s-ref` n s`)` which returns the n-th element of stream s.

L4. Suppose s_1 and s_2 are two streams of increasing numbers:

$$s_1 = (a_1\ a_2\ a_3\ a_4\ \ldots)$$
$$s_2 = (b_1\ b_2\ b_3\ b_4\ \ldots)$$

where $a_1 < a_2 < a_3 < a_4 < \ldots$ and $b_1 < b_2 < b_3 < b_4 < \ldots$
Define the operation

`(s-merge` s_1 s_2`)`

that produces the stream of elements of s_1 and s_2 in increasing order, without duplicates.

TEST: the first 20 elements of the `s-merge` of the stream of multiples of `5` with the stream of multiples of 7 is

```
> (s-take 20 (s-merge
         (s-map (lambda (x) (* 5 x)) nats)
         (s-map (lambda (x) (* 7 x)) nats)))
'(0 5 7 10 14 15 20 21 25 28 30 35 40 42 45 49 50 55 56 60)
```

L5. The stream `ham` of Hamming numbers is the stream of all numbers of the set $H = \{2^m 3^n 5^p \mid m, n, p \in \mathbb{N}\}$ enumerated in strictly increasing order. For example the first 20 elements of this stream are

```
1 2 3 4 5 6 8 9 10 12 15 16 18 20 24 25 27 30 32 36
```

Define the stream `ham`. Observe that `ham` is the stream that starts with 1, and any other Hamming number is of the form $a \cdot h$ where $a \in \{2, 3, 5\}$ and h is a Hamming number. Therefore, `(cdr ham)` is equal to the merging (with `s-merge`) of 3 streams:

(a) The stream obtained by multiplying all elements of `ham` with `2`,
(b) The stream obtained by multiplying all elements of `ham` with `3`,
(c) The stream obtained by multiplying all elements of `ham` with `5`.

TEST: The first 20 Hamming numbers:

```
> (s-take 20 ham)
'(1 2 3 4 5 6 8 9 10 12 15 16 18 20 24 25 27 30 32 36)
```

L6. Define the stream `all-primes` of all prime numbers, using the idea of the Sieve of Erathostenes:

(a) Start with the stream s of natural numbers greater or equal with 2: this is the stream `(gen-nats 2)`

(b) The stream of natural numbers is obtained by *sieving* the stream `(gen-nats 2)`. The sieving of stream s is the stream s' computed as follows:

(b1) The first element of s' is the first element of s. Let's call this element p.

(b2) Let s'' be the stream obtained by filtering out the multiples of p from the tail of s.

(b3) The tail of s' coincides with the sieving of s''.

L7. (Optional) Let $\mathtt{s1} = (a_0\ a_1\ a_2\ \ldots)$ and $\mathtt{s2} = (b_0\ b_1\ b_2\ \ldots)$ be two streams of numbers. Define `(s-prod s1 s2)` which computes the stream $c = (c_0\ c_1\ c_2\ \ldots)$ where

$$c_i = \sum_{j=0}^{i} a_j \cdot b_{i-j}$$

for all $i \in \mathbb{N}$.

SUGGESTION: Think of `s1` and `s2` as being the streams of coefficients of $x^0, x^1, x^2, \ldots$ in the infinite polynomials

$$\begin{aligned}\mathtt{pa} &= a_0 + a_1\, x + a_2 x^2 + a_3 x^3 + \ldots\\ \mathtt{pb} &= b_0 + b_1\, x + b_2 x^2 + b_3 x^3 + \ldots\end{aligned}$$

Then c is the stream of coefficients of $x^0, x^1, x^2, \ldots$ in the polynomial $\mathtt{pa} \cdot \mathtt{pb}$. Use the fact that

$$\begin{aligned}\mathtt{pa} \cdot \mathtt{pb} &= (a_0 + \mathtt{pa}_1 \cdot x)(b_0 + \mathtt{pb}_1 \cdot x)\\ &= a_0 \cdot b_0 + b_0 \cdot \mathtt{pa}_1 \cdot x + a_0 \cdot \mathtt{pb}_1 \cdot x + \mathtt{pa}_1 \cdot \mathtt{pb}_1 \cdot x^2\end{aligned}$$

where

$$\begin{aligned}\mathtt{pa}_1 &= a_1 + a_2\, x + a_3 x^2 + a_4 x^3 + \ldots\\ \mathtt{pb}_1 &= b_1 + b_2\, x + b_3 x^2 + b_4 x^3 + \ldots\end{aligned}$$

to identify a definition for the stream c.

Note that the stream of coefficients of $a_0 \cdot \mathtt{pb}_1 \cdot x$ is

```
(cons 0 (s-map (lambda (x) (* a0 x)) (s-drop 1 s2)))
```

where `a0` is the first element of stream `a`, etc.

<u>Tests</u>:

```
> (s-take 10 (s-prod all-primes all-primes))
'(4 12 29 58 111 188 305 462 679 968)
> (s-take 10 (s-prod fib fib))
'(1 2 5 10 20 38 71 130 235 420)
> (s-take 10 (s-prod all-primes fib))
'(2 5 12 24 47 84 148 251 422 702)
```

L8. Suppose `f` : $\mathbb{N} \to \mathbb{N}$ is a given function that maps natural numbers to natural numbers. Define the function

```
(s-iterate f k)
```

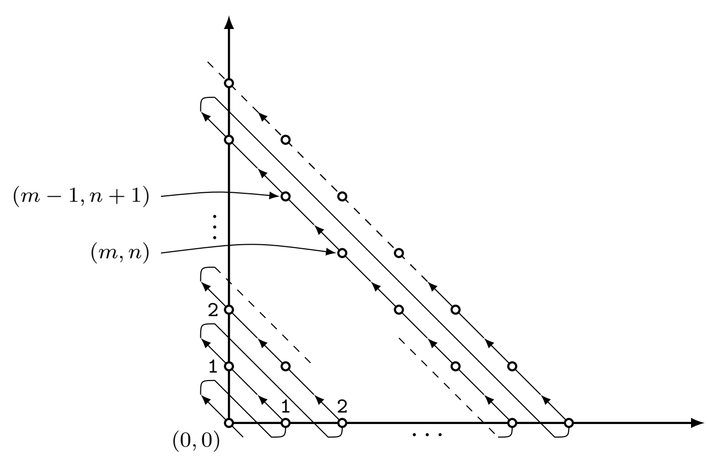
which takes as inputs such a function `f` and a number `k` $\in \mathbb{N}$, and computes a stream for the infinite list

```
k  f(k)  f(f(k))  f(f(f(k))) ...
```

<u>Tests</u>:

```
> (s-take 6 (s-iterate (lambda (x) (+ x 2))) 0)
'(0 2 4 6 8 10)
> (s-take 4 (s-iterate (lambda (x) (* x 2))) 3)
'(3 6 12 24)
```

L9. Many countable sets can be represented by streams which enumerate all their elements in a some order. For example, the elements of the set of pairs $\{(a, b) \mid a, b \in \mathbb{N}\}$ can be enumerated in the order depicted below:



In this example, the pair enumerated after (a, b) is $(a-1, b+1)$ if $a > 0$, and $(b+1, 0)$ if $a = 0$.

(a) Define the stream `nats2` of all distinct lists `'(a b)` where $a, b \in \mathbb{N}$.

(b) Define the stream of all distinct lists `'(a b)` where $a, b \in \mathbb{N}$, $a > 0, b > 0$, and a is a multiple of b.

(c) Define the stream `nats3` of all distinct lists `'(a b c)` where $a, b, c \in \mathbb{N}$.

(d) Define the stream `pithagorean` of all distinct lists `'(a b c)` where a, b, c are integers ≥ 1, and $a^2 + b^2 = c^2$.

(e) Compute the first five elements of the stream `pithagorean`.