Advanced Functional and Logic Programming

Programming principles

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Remember that ...

Recursion is a programming technique to solve problems by breaking them in one of more problems that are similar but simpler than the initial problem.

- We can define and work with
  - recursive functions, like the Fibonacci function
    ```
    (define (fib n)
       (if (= n 0) 1 (* n (fib (- n 1)))))
    ```
  - recursive datatypes, like lists of symbols:
    ```
    ⟨nlist⟩ ::= null
             | (cons ⟨symbol⟩ ⟨nlist⟩)
    ```
- In functional programming, all repetitive computations are performed by recursion
  - We can not perform usual iterative computations, because we can not change the values of variables.
  - Recursively defined functions are efficient when they are tail recursive

Structural recursion

Can be used to define functions which take one or more arguments that belong to a composite type $\langle type \rangle$:

$\langle type \rangle$ `::=` $case_1$ | ... | $case_n$

where each case $case_i$ is ($constr_i$ $\langle type_{i,1} \rangle$ ... $\langle type_{i,k_i} \rangle$)

- A recognizer of values $\text{v} \in \langle type \rangle$:

  (define ($type$? v) (or $test_1$
  ...
  $test_n$))

  where $test_i$ is of the form

  (and ($constr_i$? v) ($type_{i,1}$? ($sel_{i,1}$ v)) ... ($type_{i,k_i}$? (sel_{i,k_i} v)))

- A function with an argument $\text{v} \in \langle type \rangle$ has the form

  (define (f ... v ...)
  (cond [($constr_1$? v) ⟨computation involving ($sel_{1,1}$ v) ... (sel_{1,k_1} v)⟩]
  ...
  [($constr_n$? v) ⟨computation involving ($sel_{n,1}$ v) ... (sel_{n,k_n} v)⟩]))

Example 1: structural recursion on lists of numbers

$\langle \text{lon} \rangle$ `::= null | (cons` $\langle \text{number} \rangle$ $\langle \text{lon} \rangle$`)`

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1. Recognizer (`lon? v`):

```
(define (lon? v)
  (or (null? v)
      (and (number? (car v)) (lon? (cdr v)))))
```

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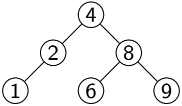
```
(define (lon? v)
  (or (null? v)
      (and (number? (car v)) (lon? (cdr v)))))
```

2. Define `(app2 lst1 lst2)` which appends the lists of numbers `ls1`, `lst2`:

```
(define (app2 lst1 lst2)
   (cond [(null? lst1) lst2]
         [(cons? lst1) (cons (car lst1)
                             (app2 (cdr lst1) lst2))]))
```

Example 2: structural recursion on binary trees of numbers

⟨nTree⟩ `::= null | (list` ⟨number⟩ ⟨nTree⟩ ⟨nTree⟩`)`



has the encoding

```
(list 4 (list 2 (list 1 null null)
                null)
        (list 8 (list 6 null null)
                (list 9 null null)))
```

1. Recognizer (nTree? v):

```
(define (nTree? v)
  (or (null? v)
      (and (list? v) (= (length v) 3)
           (number? (car v)) (nTree?  (cadr v)) (nTree?  (caddr v)))))
```

2. Define (sumNodes t) which computes the sum of numbers in all nodes of t ∈ ⟨nTree⟩:

```
(define (sumNodes t)
  (cond [(null? t) 0]
        [(list? t) (+ (car t) (sumNodes (cadr t))  (sumNodes (caddr t)))]))
```

Structural recursion on numbers

Choose the best structural description of $\mathbb{N}$

- There are many ways to define the type $\langle\text{nat}\rangle$ of natural numbers by structural recursion. For example:

  $\langle\text{nat}\rangle$ `::= 0 | (+ 1` $\langle\text{nat}\rangle$`)`

  or

  $\langle\text{nat}\rangle$ `::= 0 |` $\langle\text{even}\rangle$ `|` $\langle\text{odd}\rangle$
  $\langle\text{even}\rangle$ `::= (* 2` $\langle\text{nat}\rangle$`)`
  $\langle\text{odd}\rangle$ `::= (+ 1` $\langle\text{even}\rangle$`)`

  or . . .

- Different choices affect the efficiency of the recursive function (see next slide).

Recursive functions with numeric arguments

Example: computation of x^n for $n \in \mathbb{N}$

1. Version 1: $x^n = \begin{cases} 1 & \text{if } n = 0, \\ x \cdot x^{n-1} & \text{if } n > 0. \end{cases}$

```
(define (expt-v1 x n)
   (if (= n 0) 1 (* x (expt-v1 x (- n 1)))))
```

2. Version 2: $x^n = \begin{cases} 1 & \text{if } n = 0, \\ (x^2)^{n/2} & \text{if } n \text{ is even}, \\ x * (x^2)^{(n-1)/2} & \text{if } n \text{ is odd}. \end{cases}$

```
(define (expt-v2 x n)
   (cond [(= n 0) 1]
         [(even? n) (expt-v2 (* x x) (/ n 2))]
         [(odd? n) (* x (expt-v2 (* x x) (/ (- n 1) 2)))]))
```

Solving a more general problem

Sometimes, the best way to solve a problem is to solve a more general problem and use it to solve the original problem as a special case.

Example

Suppose `von` is a vector of numbers. Define `(vector-sum von)` which returns the sum of elements of `von`, using the functions

- `(vector-length von)`: returns the length (number of elements) of `von`
- `(vector-ref von i)`: returns the `i`-th element of `von`; the elements of `von` are indexed starting from 0.

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Instead of defining `(vector-sum von)`, we can define the **more general function** `(partial-vector-sum von n)` which computes the sum of elements with indexes from 0 to `n` in `von`.

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Instead of defining (`vector-sum von`), we can define the **more general function** (`partial-vector-sum von n`) which computes the sum of elements with indexes from 0 to `n` in `von`.

- Note that (`vector-sum von`) coincides with (`partial-vector-sum von (vector-length von)`)

Solving a more general problem

Example: compute the sum of elements of a vector of numbers

```
(define (vector-sum von)
   (define (partial-vector-sum von n)
      (if (= n 0)
         0  ; nothing to add
         (+ (partial-vector-sum von (- n 1))
             (vector-ref von (- n 1)))))
   (partial-vector-sum von (vector-length von)))
```

Remarks

1. `partial-vector-sum` is an auxiliary function used only in the implementation of `vector-sum`
   - ⇒ we can make the definition of `partial-vector-sum` local to the body of `vector-sum`
2. `von` refers to the same vector of numbers in all function calls
   - ⇒ The formal argument `von` can be removed from the definition of `partial-vector-sum`

Solving a more general problem

Example: compute the sum of elements of a vector of numbers

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  (define (partial-vector-sum von n)
    (if (= n 0)
      0  ; nothing to add
      (+ (partial-vector-sum von (- n 1))
         (vector-ref von (- n 1)))))
  (partial-vector-sum von (vector-length von)))
```

NOTE: This simplified implementation is more efficient.

Linear recursive functions

A function $f : \mathbb{N} \to \mathbb{N}$ is **linear recursive** if

- We know the first k values of f:

$$f(0) = f_0, f(1) = f_1, \ldots, f(k-1) = f_{k-1}$$

- We know $a_1, \ldots, a_k$ such that, for all $n \geq k$:

$$f(n) = a_1 \cdot f(n-1) + a_2 \cdot f(n-2) + \ldots + a_k \cdot f(n-k)$$

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All linear recursive functions have a tail recursive definition.
(see next slide)

Linear recursive functions

A tail recursive definiton

Assume $f : \mathbb{N} \to \mathbb{N}$ is **linear recursive**:

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- $f(n) = a_1 \cdot f(n-1) + a_2 \cdot f(n-2) + \ldots + a_k \cdot f(n-k)$ for all $n \geq k$

Linear recursive functions
A tail recursive definiton

Assume $f : \mathbb{N} \to \mathbb{N}$ is **linear recursive**:

- $f(0) = f_0, f(1) = f_1, \ldots, f(k-1) = f_{k-1}$
- $f(n) = a_1 \cdot f(n-1) + a_2 \cdot f(n-2) + \ldots + a_k \cdot f(n-k)$ for all $n \geq k$

Then

$$f(n) = \textit{f-acc}(n, f_0, f_1, \ldots, f_{k-1})$$

where $\textit{f-acc}(n, A_0, A_1, \ldots, A_{k-1})$ is

- A_n if $0 \leq n < k$, (the base cases)
- $\textit{f-acc}(A_1, \ldots, A_{k-1}, a_1 \cdot A_{k-1} + a_2 \cdot A_{k-2} + \ldots + a_k \cdot A_0)$, otherwise.

PROOF:

$\textit{f-acc}(n, f_0, f_1, \ldots, f_{k-2}, f_{k-1}) =$
$\textit{f-acc}(n, f(0), f(1), \ldots, f(k-2), f(k-1)) =$
$\textit{f-acc}(n-1, f(1), f(2), \ldots, f(k-1), f(k)) =$
$\ldots$
$\textit{f-acc}(k-1, f(n-k+1), f(n-k+2), \ldots, f(n-1), f(n)) =$
$f(n)$

Primitive recursive functions

A definition by primitive recursion of h : $\mathbb{N} \times A_1 \times \ldots \times A_k \to B$ looks as follows:

1. $h(0, x_1, \ldots, x_k) := f(x_1, \ldots, x_k)$ (base case)
2. $h(n+1, x_1, \ldots, x_k) := g(n, h(n, x_1, \ldots, x_k), x_1, \ldots, x_k)$ for all $n \in \mathbb{N}$. (recursive case)

Remark

Every primitive recursive function h has a tail recursive definition.

```
(define (h n x1 ... xk)
  (define (h-acc i prev)
    (if (= i n)
        prev
        (h-acc (+ i 1) (g i prev x1 ... xk))))
  (h-acc 0 (f x1 ... xk)))
```

apply and map

$\langle\text{list}\rangle$ `::= null | (cons` $\langle\text{value}\rangle$ $\langle\text{list}\rangle$`)`

- If `f` is a function and `lst` a list of values $v_1, \ldots, v_n$ then

  `(apply f lst)`

  returns the value of the function call $(f\ v_1\ \ldots\ v_n)$
- If `f` is a function which expects `n` arguments, and

$$
\begin{array}{rl}
l_1 & = (\texttt{list}\ v_{11}\ v_{12}\ \ldots\ v_{1k}) \\
\ldots & \\
l_n & = (\texttt{list}\ v_{n1}\ v_{12}\ \ldots\ v_{nk}) \\
\text{then } (\texttt{map}\ f\ l_1\ \ldots\ l_n) & = (\texttt{list}\ v_1\ \ v_2\ \ \ldots\ \ v_k) \\
\text{where } v_1 & = (f\ v_{11}\ v_{21}\ \ldots\ v_{n1}) \\
v_2 & = (f\ v_{12}\ v_{22}\ \ldots\ v_{n2}) \\
 & \ldots \\
v_k & = (f\ v_{1k}\ v_{2k}\ \ldots\ v_{nk})
\end{array}
$$

apply and map
Examples

```
> (apply + '(1 2 3 4)) ; compute 1+2+3+4
10
> (apply append '((1 2) (a b) () (3 4)))
'(1 2 a b 3 4)
> (map (lambda (x) (* x x)) '(1 2 3))
'(1 4 9)
> (map cons '(a b c) '(1 2 3))
'((a . 1) (b . 2) (c . 3))
> (map reverse '((a b c) (#t #f) ((1 2) (3 4))))
'((c b a) (#f #t) ((3 4) (1 2)))
```

REMARK: `(reverse lst)` returns the list of elements of `lst` in reverse order: It can be defined by tail recursion:

```
(define (reverse lst)
   (define (reverse-acc lst acc) ; reverse-acc is tail recursive
      (if (null? lst)
          acc
          (reverse-acc (cdr lst) (cons (car lst) acc))))
   (reverse-acc lst '()))
```

filter, foldl, foldr

1. If p is a boolean function, and lst is a list, then

   `(filter p lst)`

   returns lst without the elements v for which (p v) is #f.

2. If $f : A \times B \to B$ is a binary function, $b \in B$, and lst is a list of values $a_1, \ldots, a_n \in B$, then

   `(foldl f b lst)`

   returns the value of $(f(a_n, \ldots (f(a_1, b)) \ldots)$.
   If lst is '(), the returned value is b.

3. If $f : A \times B \to B$ is a binary function, $b \in B$, and lst is a list of values $a_1, \ldots, a_n \in B$, then

   `(foldr f b lst)`

   returns the value of $(f(a_1, \ldots (f(a_n, b)) \ldots)$.
   If lst is '(), the returned value is b.

filter,foldl,foldr

Examples

```
> (filter symbol? '(a 1 (1 . 2) bc "bc" #t ()))
'(a bc)
> (filter (lambda (x) (and (list? x)
                           (= (length x) 2)
                           (number? (car x))
                           (number? (cadr x))))
      '((1 . 2) (4 3) #(1 2) #t abc (3 2) (1 2 3)))
'((4 3) (3 2))
> (foldl (lambda (b a) (list b a)) 'a '(b1 b2 b3))
'(b3 (b2 (b1 a)))
> (foldr (lambda (b a) (list b a)) 'a '(b1 b2 b3))
'(b1 (b2 (b3 a)))
> (foldl cons '() '(a b c))
'(c b a)
> (foldr cons '() '(a b c))
'(a b c)
```

Properties of filter,foldl,foldr

filter, foldl and foldr are predefined functions.
We can define them recursively:

```
(define (filter p lst)
  (cond [(null? lst) null]
        [(p (car lst)) (cons (car lst) (filter p (cdr lst)))]
        [#t (filter p (cdr lst))]))
(define (foldl f b lst)
   (cond [(null? lst) b]
         [#t (foldl f (f (car lst) b) (cdr lst))]))
(define (foldr f b lst)
   (cond [(null? lst) b]
         [#t (f (car lst) (foldr f b (cdr lst)))]))
```

1. foldl is tail recursive → efficient implementation
2. foldr is not tail recursive, but note that:
   - (foldr f b lst)=(foldl f b (reverse lst))

   ⇒ a better definition of foldr, which is tail-recursive:

   ```
   (define (foldr f b lst) (foldl f b (reverse lst))
   ```

Remarks

Several real-world applications operate on large collections of data that can be implemented as lists, and the operations on such collections are compositions of a small number of generic operations.

- This observation led to the idea of applicative programming with lists, where all operations of practical interest are defined as combinations of a small number of generic operations, such as

  ```
  map
  apply
  filter
  foldl
  foldr
  reverse
  ```

- Applicative programming with lists inspired the MapReduce programming model used by Google to process and generate large data sets.

Illustrated example

A database of employees

Database: list of records of the form `(list` *name salary job*`)`

```
> (define employees
'(("John White" 10000 "manager") ("Sam Smith" 4500 "cook")
  ("Alice Cooper" 3600 "secretary") ("Ray Ban" 6320 "driver")
  ("Mike Cole" 2600 "waiter") ("Ana Fox" 3800 "secretary")
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O_1) Get the records of people with salary greater that 4000:

```
> (filter (lambda (emp) (> (cadr emp) 4000)) employees)
'(("John White" 10000 "manager")
  ("Sam Smith" 4500 "cook")
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O_2) Get a digest with the names of all people employed in the company

```
> (map car employees)
'("John White" "Sam Smith" "Alice Cooper" "Ray Ban"
  "Mike Cole" "Ana Fox" "John Black" "Jack White")
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```

O_3) Total amount of money spent to pay the employees' salaries:

```
> (foldl (lambda (emp b) (+ (cadr emp) b)) 0 employees)
36770
```

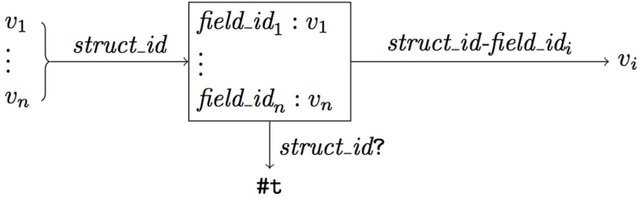
The struct special form

Users can define their own composite types with the special form struct:

(struct *struct_id* (*field_id*$_1$ ... *field_id*$_n$))

creates the following functions for the newly declared type *struct_id*:

- the **constructor** *struct_id* that takes as many arguments as the number of fields
- the **recognizer** *struct_id*?
- n **selectors** *struct_id*–*field_id*$_i$, for each of the fields of the new composite type



Structures
Examples

```
> (struct emp (name salary job))
> ; create an instance of emp
  (define e1 (emp "Ana Schwarz" 2300 "attorney"))
> (emp? e1)
#t
> (list (emp-name e1) (emp-salary e1))
'("Ana Schwarz" 2300)
```

- By default, the values of structures are opaque:
  ```
  > e1
  #<emp>
  ```
- We can define structures with transparent values if we use the `#:transparent` keyword.

Example

```
> (struct emp (name salary job) #:transparent)
> (define e2 (emp "Bruce Willis" 25000 "actor"))
> e2
(emp "Bruce Willis" 25000 "actor")
```

A convenient abbreviation

Many datatypes (including pairs, lists, vectors) have quoted values of the form *'datum*

Usually, these forms are valid input forms.

Examples

```
> '(1 2 (3 4))   ; shorter input than (list 1 2 (list 3 4))
'(1 2 (3 4))
> '#(1 2 (3 4)) ; shorter input than (vector 1 2 (list 3 4))
'#(1 2 (3 4))
>'(a . #(b c))   ; shorter input than (cons 'a (vector 'b 'c))
'(a . #(b c))
```

Bad news: quoted expressions **can not** be used to create composite values from component values.

Good news: quasiquoted expressions **can** be used to create composite values from component values (see next slide).

Quasiquoted expresions

- A **quasiquoted expression** is of the form

  ` *datum*

  It is like a quoted expression, but is starts with the character `

- Inside a quasiquoted expression, every subexpression of the form

  , *expr*

  is replaced by the value of *expr*

- Inside a quasiquoted expression, every subexpression of the form

  ,@*expr*

  where the value of *expr* is a list of values $v_1, \ldots, v_n$, is replaced by the sequence of values $v_1 \ \ldots \ v_n$

Quasiquoted expressions
Examples

```
> (define a '(Toyota Prius))
> (define b 2011)
> (define c "red")
> ; a quasiquoted list
  `(car (model ,@a) (year ,b) (color ,c))
'(car (model Toyota Prius) (year 2011) (color "red"))
> `#(,@a is ,c) ; a quasiquoted vector
'#(Toyota Prius is "red")
```

Quasiquoted expressions
Examples

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> (define b 2011)
> (define c "red")
> ; a quasiquoted list
  `(car (model ,@a) (year ,b) (color ,c))
'(car (model Toyota Prius) (year 2011) (color "red"))
> `#(,@a is ,c) ; a quasiquoted vector
'#(Toyota Prius is "red")
```

Question

What is the following function doing when `lst` is a list of integers?

```
(define (del2 lst)
  (if (null? lst) '()
      (if (even? (car lst))
          `(,@(del2 (cdr lst)))
          `(,(car lst) ,@(del2 (cdr lst))))))
```

A general pattern of recursion

This is a frequent pattern to transform composite values:

```
(define (transform cv)
  (cond [base-case₁ value₁]
        ...
        [base-caseₘ valueₘ]
        [recursive-case₁
          ; compute (transform v₁) ... (transform vₙ)
          ; for the component values v₁, ..., vₙ of cv
          ; and combine them into a return value
        ]
        [recursive-case₂ ...]
        ...))
```

Example

Compute the flattened form of a nestes list of symbols sl $\in \langle \mathrm{S}-\mathrm{list}\rangle$

$$\langle \mathrm{S}-\mathrm{list}\rangle ::= \langle \mathrm{symbol}\rangle \mid (\texttt{list}\ \langle \mathrm{S}-\mathrm{list}\rangle\ \ldots\ \langle \mathrm{S}-\mathrm{list}\rangle)$$

The flattened form of an S-list sl is defined as follows:

- (list sl) if sl is a symbol.
- Otherwise, sl $=$ (list $\mathtt{sl}_1\ \ldots\ \mathtt{sl}_n$) and the flattened form is the result of appending the flattened forms of $\mathtt{sl}_1, \ldots, \mathtt{sl}_n$ in this order.

Example

```
> (flatten-list '(((a) (b (() c d)) e f ((g)))))
'(a b c d e f g)
```

Example

Compute the flattened form of a nestes list of symbols `sl` $\in \langle \mathrm{S-list} \rangle$

$\langle \mathrm{S-list} \rangle$::=$\langle \mathrm{symbol} \rangle$ | (list $\langle \mathrm{S-list} \rangle$ ... $\langle \mathrm{S-list} \rangle$)

The flattened form of an S-list `sl` is defined as follows:

- (list sl) if `sl` is a symbol.
- Otherwise, $\mathtt{sl} = (\mathtt{list}\ \mathtt{sl}_1\ \ldots\ \mathtt{sl}_n)$ and the flattened form is the result of appending the flattened forms of $\mathtt{sl}_1, \ldots, \mathtt{sl}_n$ in this order.

Example

```
> (flatten-list '(((a) (b (() c d)) e f ((g)))))
'(a b c d e f g)
```

Implementation

```
(define (flatten-list sl)
  (cond [(symbol? sl) (list sl)]
        [#t (apply append (map flatten-list sl))]))
```

An illustrated example

Inward propagation of negation in propositional formulas

```
⟨prop⟩ ::= ⟨symbol⟩                      ; atomic formula
         | (list 'not ⟨prop⟩)            ; negation
         | (list 'and ⟨prop⟩ ⟨prop⟩)     ; conjunction
         | (list 'or  ⟨prop⟩ ⟨prop⟩)     ; disjunction
```

- If cv $= \neg(P \vee Q)$ then $\mathit{transf}(\mathtt{cv}) = \mathit{transf}(\neg P) \wedge \mathit{transf}(\neg Q)$.
- If cv $= \neg(P \wedge Q)$ then $\mathit{transf}(\mathtt{cv}) = \mathit{transf}(\neg P) \vee \mathit{transf}(\neg Q)$.
- If cv $= \neg(\neg P)$ then $\mathit{transf}(\mathtt{cv}) = \mathit{transf}(P)$.
- If cv is an atom, then $\mathit{transf}(\mathtt{cv}) = \mathtt{cv}$.
- If cv is $P \vee Q$ then $\mathit{transf}(\mathtt{cv}) = \mathit{transf}(P) \vee \mathit{transf}(Q)$.
- If cv is $P \wedge Q$ then $\mathit{transf}(\mathtt{cv}) = \mathit{transf}(P) \wedge \mathit{transf}(Q)$.
- If cv is $\neg P$ where P is not negation, then $\mathit{transf}(\mathtt{cv}) = \neg \mathit{transf}(Q)$.

Top down transformation of composite values

Example: inward propagation of negation in propositional formulas (contd.)

- Useful recognizers for all kinds of propositional formulas:

```
(define (atom? f)
    (symbol? f))
(define (not? f)
   (and (list? f) (= (length f) 2) (eq? (car f) 'not)))
(define (and? f)
   (and (list? f) (= (length f) 3) (eq? (car f) 'and)))
(define (or? f)
   (and (list? f) (= (length f) 3) (eq? (car f) 'or)))
```

Top down transformation of composite values

Example: inward propagation of negation in propositional formulas (contd.)

- Definition of (propagate-not cv)

```
(define (propagate-not cv)
  (cond [(atom? cv) cv]
        [(and (not? cv) (or? (cadr cv)))
         (let ([P (list-ref (cadr cv) 1)]
               [Q (list-ref (cadr cv) 2)])
           `(and ,(propagate-not `(not ,P))
                 ,(propagate-not `(not ,Q))))]
        [(and (not? cv) (and? (cadr cv)))
         (let ([P (caddr cv)]
               [Q (cadddr cv)])
           `(or ,(propagate-not `(not ,P))
                ,(propagate-not `(not ,Q))))]
        [(and (not? cv) (not? (cadr cv)))
         (propagate-not (list-ref (cadr cv) 1))]
        [#t `(,(car cv) ,@(map propagate-not (cdr cv)))]))
```