String matching

The finite automaton approach. The Aho-Corasick algorithm

November 8, 2019

- An alphabet Σ is a finite set of characters.
- A string S of length $n \geq 0$ is an array $S[1..n]$ of characters from Σ. We write $|S|$ for the length of S. Thus, $|S| = n$
- $S[i]$ is the character of S at position i
- $S[i..j]$ represents the substring of S form position i to position j inclusively.

Example

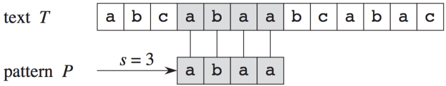
If $S = \texttt{alphabet}$ then $|S| = 8$, $S[1] = \texttt{a}$, $S[2] = \texttt{b}$, $S[1..4] = \texttt{alph}$, $S[3..7] = \texttt{phabe}$

ASSUMPTIONS:

- Σ : finite set of characters (an alphabet).
  E.g., $\Sigma = \{a, b, \ldots, z\}$
- $P[1..m]$: array of $m > 0$ characters from Σ (the pattern)
- $T[1..n]$: array of $n > 0$ characters from Σ (the text)

We say that P occurs with shift s in T (or, equivalently, that P occurs beginning at position $s+1$ in T) if $0 \leq s \leq n-m$ and $T[s+1..s+m] = P[1..m]$ (that is, if $T[s+j] = P[j]$, for $1 \leq j \leq m$).

EXAMPLE:



The string matching problem

Given a pattern $P[1..m]$ and a text $T[1..n]$

Find all shifts s where P occurs in T.

Terminology and notation:

- Σ^*=the set of all strings of characters from Σ
- If $x, y \in \Sigma^*$ then
  - $x\,y$:=the concatenation of x with y
  - $|x| :=$ the length (number of characters) of x
  - ϵ :=the zero-length empty string
  - x is prefix of y, notation $x \sqsubseteq y$, if $y = x\,w$ for some $w \in \Sigma^*$.
    x is suffix of y, notation $x \sqsupseteq y$, if $y = w\,x$ for some $w \in \Sigma^*$.

  Example: $\underline{\texttt{ab}} \sqsubseteq \underline{\texttt{ab}}\texttt{cca}$

REMARKS

1. $x \sqsupseteq y$ if and only if $x\,a \sqsupseteq y\,a$.
2. Every string is either ϵ, or of the form wa where $a \in \Sigma$ and w a string.

The naive string matching algorithm

NAIVESTRINGMATCHER(T, P)

$n := T.length$

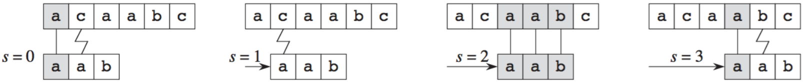
$m := P.length$

for $s = 0$ **to** $n - m$

    if $P[1..m] == T[s+1..s+m]$

        print "pattern occurs with shift" s

EXAMPLE:



(a) (b) (c) (d)

- Time complexity: $O((n - m + 1)\,m)$
  - Several character comparison are performed repeatedly
  - **Can we do better?**

String matching with finite automata

Definition (Finite automaton)

A finite automaton is a 5-tuple $\mathcal{A} = (Q, q_0, A, \Sigma, \delta)$ where

- Q : finite set of states
- $q_0 \in Q$: the start state
- $A \subseteq Q$: distinguished set of accepting states
- Σ:=finite set of characters (the input alphabet)
- $\delta : Q \times \Sigma \rightarrow Q$ is the transition function

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Alternative representations of a finite automaton:

1. Tabular representation of δ
2. state-transition diagram

(see next slide)

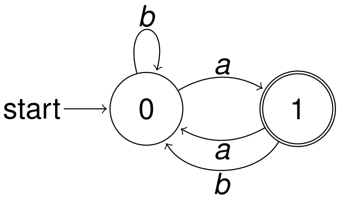
Alternative representations of a finite automaton

$\mathcal{A} = (Q, q_0, A, \Sigma, \delta)$ where
$Q = \{0, 1\}, q_0 = 0, A = \{1\}, \Sigma = \{a, b\}$

- Tabular representation:

δ	a	b
$\rightarrow 0$	1	0
$\leftarrow 1$	0	0

- State-transition diagram:



Acceptance by finite automata

ASSUMPTION: $\mathcal{A} = (Q, q_0, A, \Sigma, \delta)$ is a finite automaton.

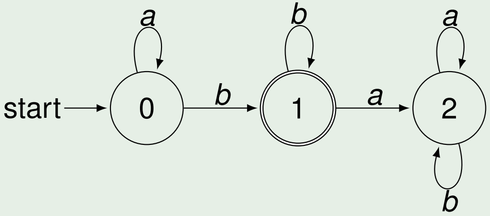
- Define inductively $\phi : \Sigma^* \to Q$, as follows:

$$\phi(\epsilon) := q_0,$$
$$\phi(wa) := \delta(\phi(w), a).$$

We say that w is accepted by $\mathcal{A}$ if $\phi(w) \in A$.

Example

The following finite automaton accepts all (and only) words of the form $a^m b^n$ where $m \geq 0$, $n \geq 1$:



REMARK: The time complexity of computing $\phi(w)$ is $O(n)$ where $n = |w|$.

- Define a finite automaton $\mathcal{A}$ such that $T[1..i]$ is accepted by $\mathcal{A}$ if and only if it has suffix P (that is, $P \sqsupseteq T[1..i]$).
- $\mathcal{A}$ can be defined in a preprocessing step of $P[1..m]$
  - To understand the construction of $\mathcal{A}$, we shall define the **suffix function** σ corresponding to pattern P:

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Definition

The suffix function corresponding to pattern $P[1..m]$ is the function $\sigma : \Sigma^* \to \{0, \ldots, m\}$ such that $\sigma(x)$ is the length of the longest prefix of P that is also a suffix of x. Formally:

$$\sigma(x) := \max\{k \mid 0 \leq k \leq m \text{ and } P[1..k] \sqsupseteq x\}.$$

A finite automaton for the string matching problem
Main ideas

- Define a finite automaton $\mathcal{A}$ such that $T[1..i]$ is accepted by $\mathcal{A}$ if and only if it has suffix P (that is, $P \sqsupseteq T[1..i]$).
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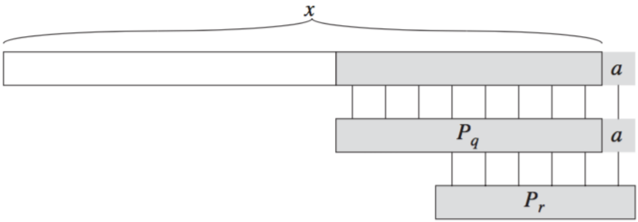
EXAMPLES: If $P = \mathtt{ab}$ then $\sigma(\epsilon) = 0$, $\sigma(\mathtt{ccac\underline{a}}) = 1$, $\sigma(\mathtt{ac\underline{ab}}) = 2$.

Suffix-function recursion lemma

For any string x and character $a \in \Sigma$, if $q = \sigma(x)$, then $\sigma(x\,a) = \sigma(P[1..q]\,a)$.

A graphical illustration of a proof of this Lemma is shown below:



The finite automaton corresponding to a pattern

ASSUMPTION: $P[1..m]$ is the given pattern,

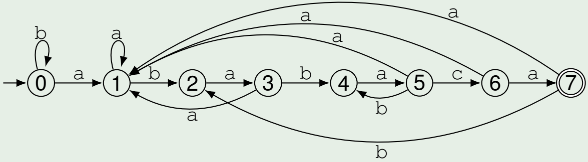
The corresponding finite automaton is $\mathcal{A} = (Q, q_0, A, \Sigma, \delta)$ where:

- $Q = \{0, 1, 2, \ldots, m\}$
- $q_0 = 0$
- $A = \{m\}$

$\delta(q, a) = \sigma(P[1..q]\, a)$

Example

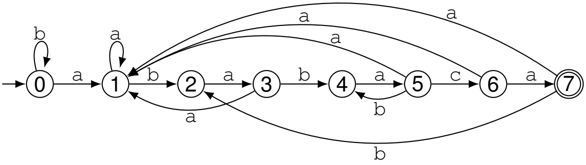
The finite automaton corresponding to $P[1..7] =$ `ababaca` is



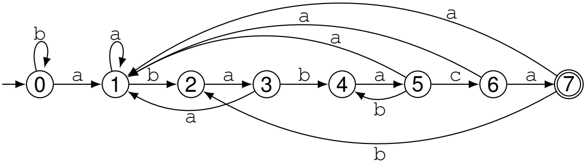
The missing transitions from a node point to state 0.

The finite automaton corresponding to a pattern

Illustrated example



i	—	1	2	3	4	5	6	7	8	9	10	11
$T[i]$	—	a	b	a	b	a	b	a	c	a	b	a
state $\phi(T_i)$	0	1	2	3	4	5	4	5	6	**7**	2	3



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The remaining question is:

How to compute the state transition function δ of $\mathcal{A}$?

Computing the transition function

A naive implementation (pseudocode)

```
ComputeTransitionFunction(P, Σ)
m := P.length
for q := 0 to m
    for each character a ∈ Σ
        k := min(m, q + 1) + 1
        repeat
            k := k − 1
        until P[1..k] ⊐ P[1..q] a
        δ(q, a) := k
return δ
```

Time complexity: $O(m^3 |\Sigma|)$.

There are better algorithms, which can compute δ with time complexity $O(m |\Sigma|)$.

We assume given

- $T[1..m]$ called text
- A finite set of patterns $\mathcal{P} = \{P_1, P_2, \ldots, P_z\}$

Find **all** positions where some $P \in \mathcal{P}$ occurs in T.

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USEFUL AUXILIARY NOTIONS

1. keyword tree $\mathcal{K}$ of the set $\mathcal{P}$
2. failure links between the nodes of $\mathcal{K}$

The keyword tree of a set of patterns $\mathcal{P} = \{P_1, \ldots, P_z\}$ is a tree $\mathcal{K}$ which satisfies 3 conditions:

1. every edge is labeled with exactly 1 character.
2. Distinct edges which leave from a node are labeled with distinct characters.
3. Every pattern $P_i \in \mathcal{P}$ gets mapped to a unique node v of $\mathcal{K}$ as follows: the string of characters along the branch from root to node v is P_i, and every leaf node of $\mathcal{K}$ is the mapping of a pattern from $\mathcal{P}$.

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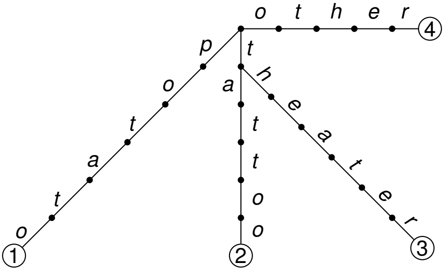
NOTATION: for every node $v \in \mathcal{K}$, $\mathcal{L}(v)$ is the string of characters along the branch of $\mathcal{K}$ from root to node v.

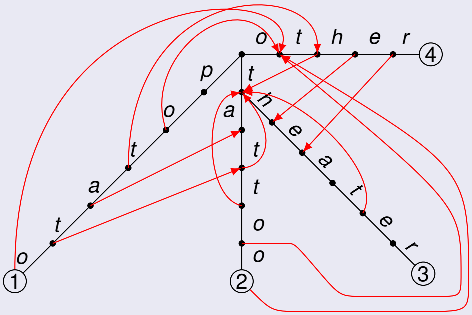
1. Keyword tree

Example for $\mathcal{P} = \{potato, tattoo, theater, other\}$



Let $\mathcal{K}$ be the keyword tree for $\mathcal{P} = \{P_1, \ldots, P_z\}$. Every node v of $\mathcal{K}$ has only one failure link to the node n_v of $\mathcal{K}$ which has the following property: $\mathcal{L}(n_v)$ is the longest proper suffix of $\mathcal{L}(v)$ which is a prefix of a pattern from $\mathcal{P}$.

Example for $\mathcal{P} = \{potato, tattoo, theater, other\}$



the failure links which are not depicted, go to the root of $\mathcal{K}$

Aho-Corasick algorithm

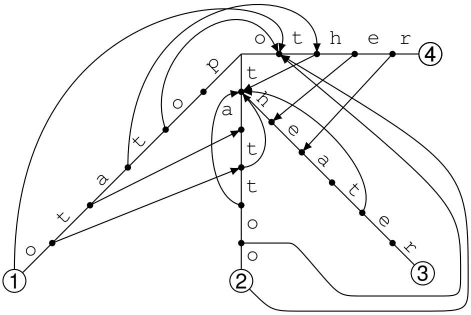
Allows to find all occurrences of $\mathcal{P}$ in $T[1..m]$ in time $O(m)$. It relies on the keyword tree $\mathcal{K}$ for $\mathcal{P}$ and its failure links.

The characters of $T[1..m]$ are read from left to right:

1. crt :=root of $\mathcal{K}$
   $i := 1$
2. If $\mathcal{L}(crt) = P_j$ or there is a sequence of failure links $crt \to \ldots \to w$ with $\mathcal{L}(w) = P_j$
   - signal "P_j occurs at position i in T"
3. If $i = m$ then STOP.
4. If $T[i] = c$ and there is an edge $crt \overset{c}{-} v$ then $i := i + 1, crt := v$, goto 2.
5. If $T[i] = c$ and there is no edge $crt \overset{c}{-} v$ then let $crt \to \ldots \to v$ the shortest sequence of failure links such that $\exists v \overset{c}{-} w$ an let $crt := v$.
   If no such sequence exists, let $crt :=$ root of $\mathcal{K}$.
6. goto 2.

Aho-Corasick algorithm

Illustrated example: $\mathcal{P} = \{potato, tattoo, theater, other\}$, $T = potheater$



potheater

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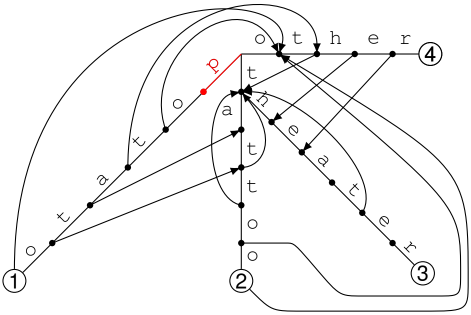


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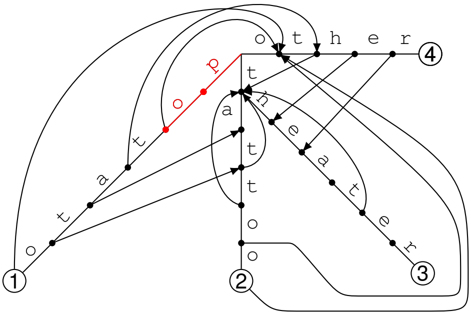


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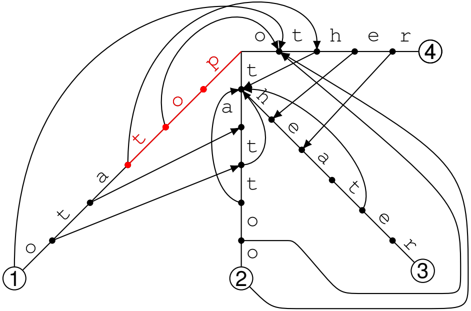


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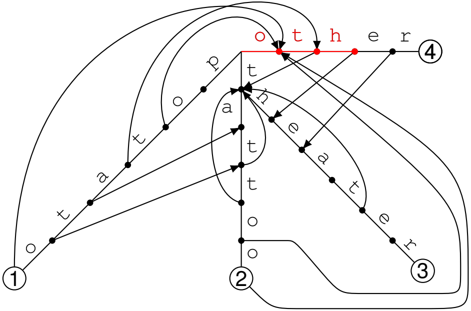


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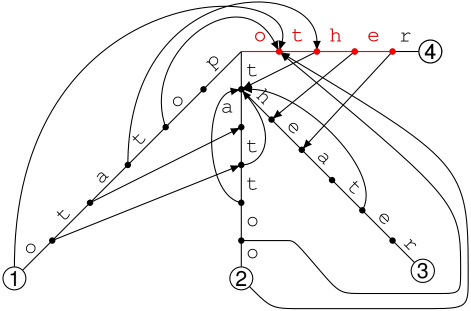


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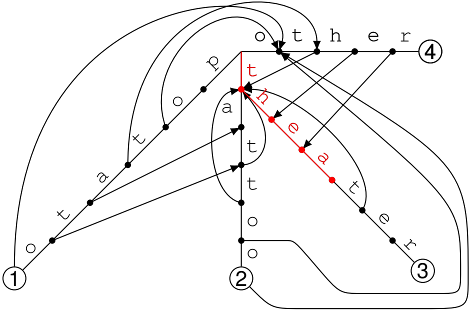


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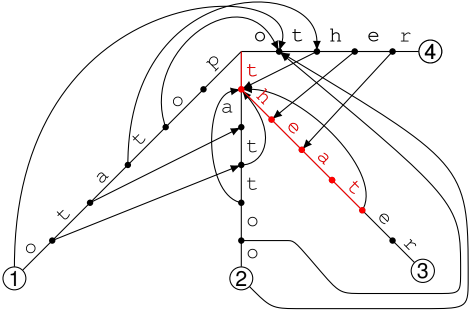


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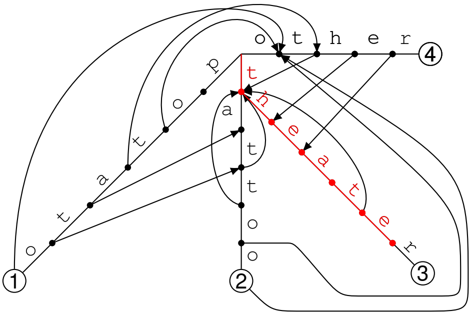


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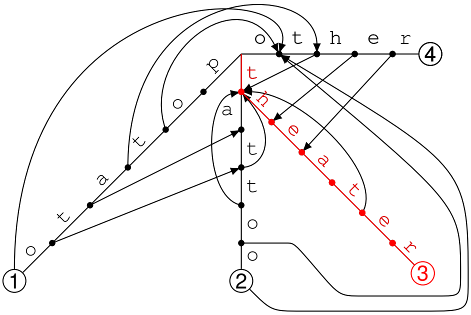


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$\Rightarrow$ detected occurrence of $P_3 =$ theater

$\mathcal{P} = \{P_1, \ldots, P_z\}$, $n := |P_1| + \ldots + |P_z|$

- The keyword tree $\mathcal{K}$ for $\mathcal{P}$ is built by adding repeatedly the edges for $P_1, \ldots, P_z$ to an initially empty tree.
  - The addition of the edges for P_i has runtime complexity $O(|P_i|)$

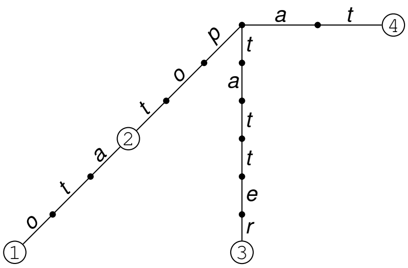
  $\Rightarrow$ the construction of $\mathcal{K}$ has runtime complexity $O(|P_1| + \ldots + |P_z|) = O(n)$
- The failure links are added to each node of $\mathcal{K}$ in the order of a breadth-first traversal: If r is the root of $\mathcal{K}$ then
  - add a failure link for the root of $\mathcal{K}$: $r \to r$
  - for the nodes of v at tree depth 1: add failure links $v \to r$
  - if v is a node at depth $k > 1$, then let
    - v' be the parent of v
    - x be the label of $v - v'$
    - $\pi : v' \to v_1 \to \ldots v_i$ be the shortest sequence of failure links such that there is an edge $v_i - w$ in $\mathcal{K}$ with label x

    If π exists: add the failure link $v \to w$

    If π does not exist: add the failure link $v \to r$

Addition of failure links to a keyword tree

Illustrated example for the keyword tree of $\mathcal{P} = \{potato, pot, tatter, at\}$



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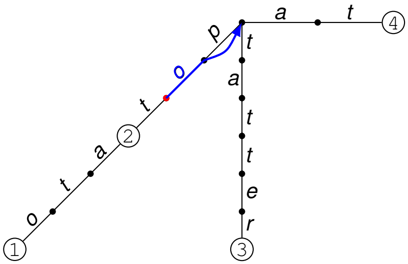

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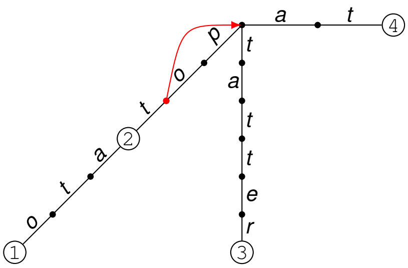

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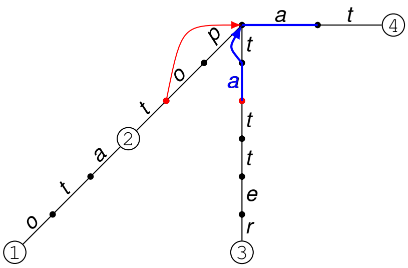

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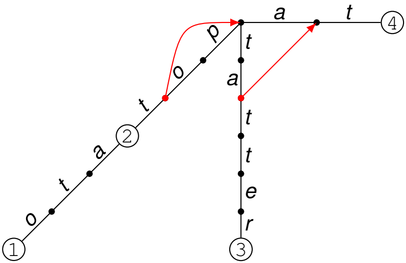

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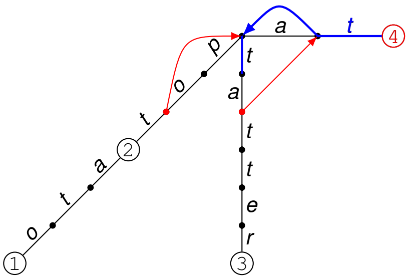

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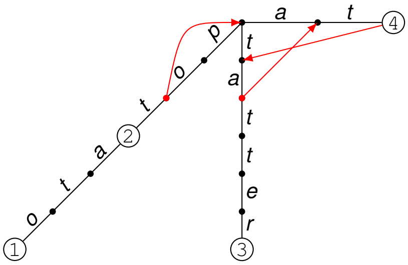

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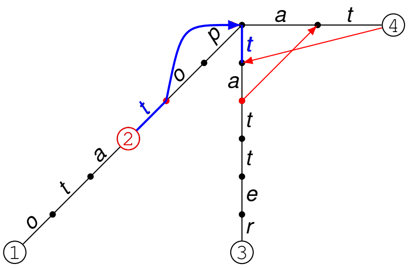

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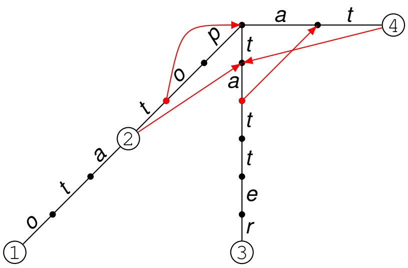

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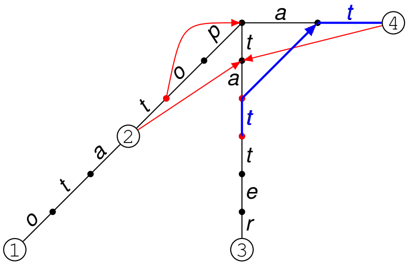

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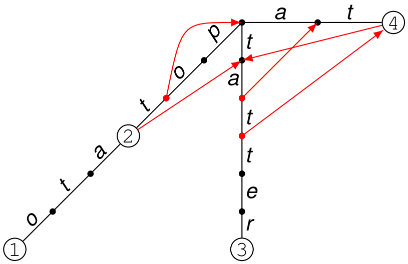
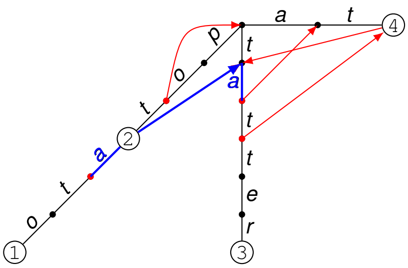


Addition of failure links to a keyword tree

Illustrated example for the keyword tree of $\mathcal{P} = \{potato, pot, tatter, at\}$



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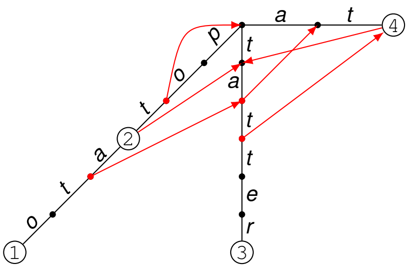

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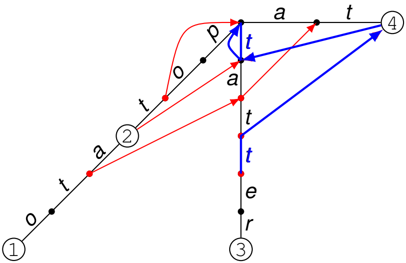

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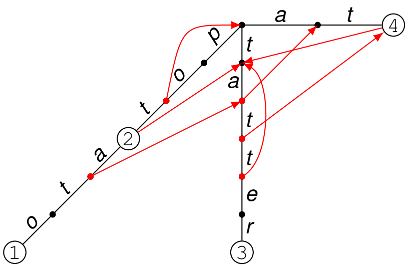

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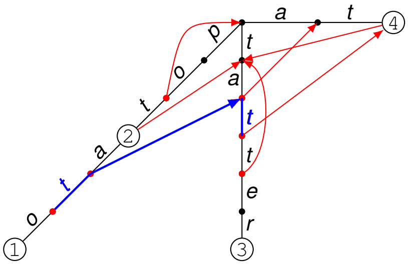

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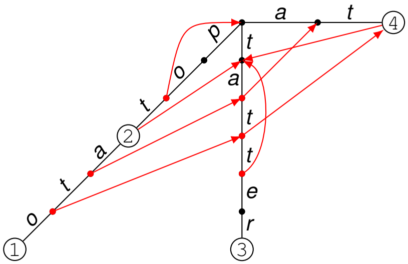

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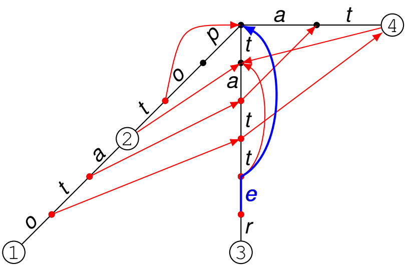

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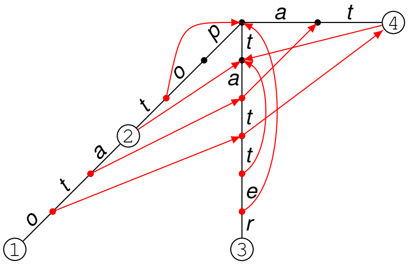

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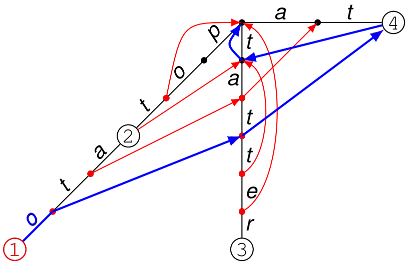

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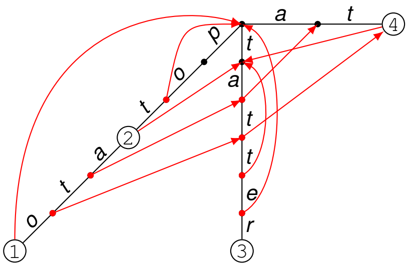

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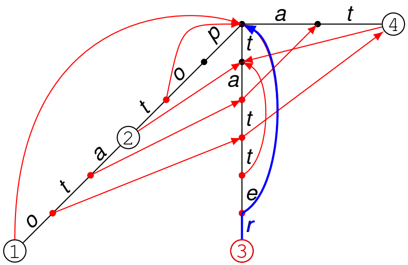

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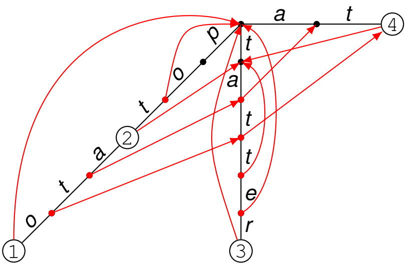

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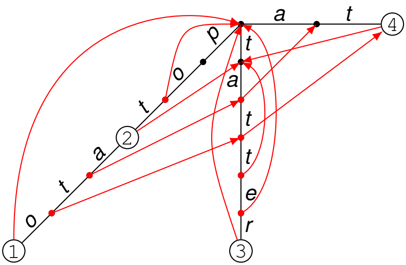

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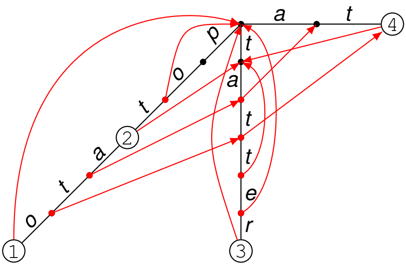

REMARK: The runtime complexity of this algorithm for the computation of failure links is $O(n)$, where $n = |P_1| + \ldots + |P_z|$

- A proof of this fact can be found in the recommended bibliography.

References

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- D. Gusfield: *Algorithms on Strings, Trees, and Sequences*. Published by *Press Syndicate of the University of Cambridge*. 1997.