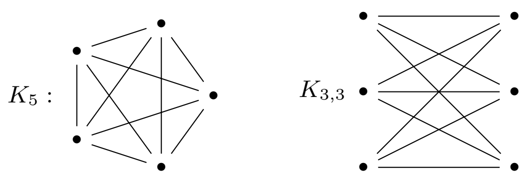
Exercises related to lecture 12

This lecture is about planar graphs, Euler Theorem, graph colourings, and chromatic polynomials.

You are expected to know the statement and the proof of Euler Theorem, and some if its consequences that can be used to check if a graph is planar or not.

1. Draw the graphs K_5 and $K_{3,3}$.

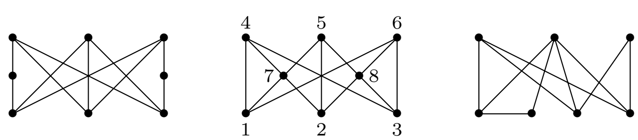
   Answer



2. What is a subdivision of a graph?

3. What does Kuratowski's Theorem say?

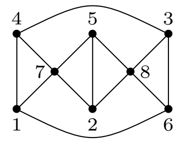
4. Which of the following graphs is planar and which is not? Which one is a subdivision of $K_{3,3}$ and why?



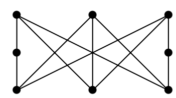
G_1 G_2 G_3

   Answer:

   - G_1 is not planar because it obviously contains a subdivision of $K_{3,3}$.
   - G_2 contains no subdivision of K_5 because it has only 2 nodes with degree ≥ 4. It can also be argued that G_2 contains no subdivision of $K_{3,3}$. The following is a planar representation of G_2 (note that I swapped the positions of nodes 3 and 6):

G_2 

- G_3 is not planar because it contains a subdivision of $K_{3,3}$: We can reposition the nodes of G_3, such that it looks as follows:

G_3 

5. Prove that every planar graph is 5-colorable.

6. What is the statement of the 4-colour map colouring problem?

You should be able to compute the chromatic polynomial of some simple graphs.

1. Find a general formula for the chromatic polynomial $c_{C_n}(z)$ when C_n is a cycle with n nodes. What is the chromatic number of C_n?

   ANSWER: Let $a_n = c_{C_n}(z)$.

   If $n = 3$ then $C_n = K_3$ and $a_3 = c_{K_3}(z) = z(z-1)(z-2)$.

   If $n > 3$, let e be some edge of C_n. It is obvious that $C_n - e = P_n$ and $C_n/e = C_{n-1}$. Therefore

$$\begin{aligned} a_n &= c_{C_n}(z) = \chi_{P_n}(z) - \chi_{C_{n-1}}(z) = z \cdot (z-1)^{n-1} - a_{n-1} \\ &= (z-1)^n + (z-1)^{n-1} - a_{n-1} \end{aligned}$$

   Thus $\underbrace{a_n - (z-1)^n}_{b_n} = (-1)\underbrace{\left(a_{n-1} - (z-1)^{n-1}\right)}_{b_{n-1}}$. Let $b_n = a_n - (z-1)^n$.

   From $b_n = -b_{n-1}$ we learn that $b_n = (-1)^{n-1} b_1 = (-1)^2 b_{n-1} = \ldots = (-1)^{n-3}\, b_{n-3} = (-1)^{n-3} \cdot (a_3 - (z-1)^3) = (-1)^{n-3}(z(z-1)(z-2) - (z-1)^3) = (-1)^{n-3}(z-1)(z\,(z-2) - (z-1)^2) = (-1)^{n-2}(z-1)$. Thus

$$\begin{aligned} c_{C_n}(z) = a_n &= (z-1)^n + b_n = (z-1)^n + (-1)^{n-2}(z-1) \\ &= (z-1)\left((z-1)^{n-1} + (-1)^{n-2}\right) \end{aligned}$$

   The chromatic number $\chi(C_n)$ is the least $k > 0$ such that $\chi_{C_n}(k) > 0$.

   - If $n = 3$ then $c_{C_3}(z) = z(z-1)(z-2)$ and $\chi(C_3) = 3$.
   - If $n > 3$ is even ($n = 2 \cdot m$ with $m \geq 2$) then $\chi(C_n) = 2$ because

$$\begin{aligned} c_{C_n}(1) &= (1-1)\left((1-1)^{2\,m-1} + (-1)^{2\,m-2}\right) = 0 \\ c_{C_n}(2) &= (2-1)\left((2-1)^{2\,m-1} + (-1)^{2\,m-2}\right) = 2 > 0 \end{aligned}$$

- If $n > 3$ is odd ($n = 2 \cdot m + 1$ with $m \geq 2$) then $\chi(C_n) = 3$ because

$$\begin{aligned} c_{C_n}(1) &= (1-1)\left((1-1)^{2m} + (-1)^{2m-1}\right) = 0 \\ c_{C_n}(2) &= (2-1)\left((2-1)^{2m} + (-1)^{2m-1}\right) = 1 \cdot (1-1) = 0 \\ c_{C_n}(3) &= (3-1)\left((3-1)^{2m} + (-1)^{2m-1}\right) = 2 \cdot (2^{2m} - 1) > 0 \end{aligned}$$

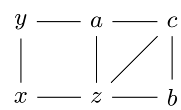
2. How many 2-colorings has a tree with n nodes?

   Answer

   From the lecture notes, we know that $c_{T_n}(z) = z \cdot (z-1)^{n-1}$ for any tree T_n with n nodes. The number of 2-colorings of T_n is

$$c_{T_n}(2) = 2 \cdot (2-1)^{n-1} = 2 \cdot 1^{n-1} = 2.$$

3. Let G be the graph depicted below



   (a) Draw the graphs $G - (a, z)$ and $G/(a, z)$.

   (b) Compute the chromatic polynomial of G.

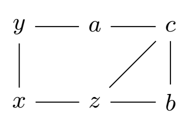
   (c) How many 2-colourings has G?

   (d) How many 3-colourings has G?

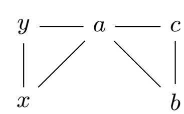
   (e) What is the chromatic number of G?

   Answer

   (a) $G_1 = G - (a, z)$:

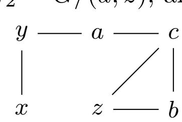


   $G_2 = G/(a, z)$:

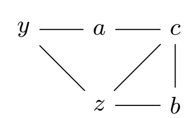


   (b) $c_G(z) = c_{G_1}(z) - c_{G_2}(z) = c_{G_3}(z) - c_{G_4}(z) - (c_{G_5}(z) - c_{G_6}(z))$, where $G_1 = G - (a, z)$, $G_2 = G/(a, z)$, and

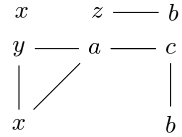
   $G_3 = G_1 - (x, z)$:



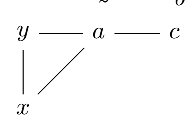
   $G_4 = G_1/(x, z)$:



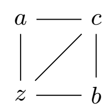
   $G_5 = G_2 - (a, b)$:



   $G_6 = G_2/(a, b)$:



$$
\begin{aligned}
c_{G_3}(z) &= c_{G_3-(b,c)}(z) - c_{G_3/(b,c)}(z) = c_{P_6}(z) - c_{P_5}(z)\\
&= z(z-1)^5 - z(z-1)^4\\
c_{G_4}(z) &= c_{G_4-(a,c)}(z) - c_{G_4/(a,c)}(z) = c_{G_5}(z) - c_{G_7}(z)
\end{aligned}
$$

where $G_7 = G_4/(a,c)$: 

$$
\begin{aligned}
c_{G_5}(z) &= c_{G_5-(a,x)}(z) - c_{G_5/(a,x)}(z) = c_{P_5}(z) - c_{P_4}(z)\\
&= z(z-1)^4 - z(z-1)^3\\
c_{G_6}(z) &= c_{G_6-(a,x)}(z) - c_{G_6/(a,x)}(z) = c_{P_4}(z) - c_{P_3}(z)\\
&= z(z-1)^3 - z(z-1)^2\\
c_{G_7}(z) &= c_{G_7-(a,z)}(z) - c_{G_7/(a,z)}(z) = c_{G_6}(z) - c_{K_3}(z)\\
&= z(z-1)^3 - z(z-1)^2 - z(z-1)(z-2)
\end{aligned}
$$

Finally, we obtain

$$
\begin{aligned}
c_G(z) =& z(z-1)^5 - z(z-1)^4 - (z(z-1)^4\\
& - z(z-1)^3 - z(z-1)^3 + z(z-1)^2 + z(z-1)(z-2))\\
& - (z(z-1)^4 - z(z-1)^3 - z(z-1)^3 + z(z-1)^2)\\
=& z(z-1)^5 - 3\,z(z-1)^4 + 4\,z(z-1)^3 - 2\,z(z-1)^2 - z(z-1)(z-2)\\
=& z(z-1)((z-1)^4 - 3(z-1)^3 + 4(z-1)^2 - 2(z-1) - z + 2)\\
=& z(z-1)(z^4 - 7\,z^3 + 19\,z^2 - 24\,z + 12)\\
=& z^6 - 8\,z^5 + 26\,z^4 - 43\,z^3 + 36\,z^2 - 12\,z
\end{aligned}
$$

(c) The number of 2 colorings of G is $c_G(2) = 2(2-1)(2^4 - 7\cdot 2^3 + 19\cdot 2^2 - 24\cdot 2 + 12) = 0$.

(d) The number of 3 colorings of G is $c_G(3) = 3(3-1)(3^4 - 7\cdot 3^3 + 19\cdot 3^2 - 24\cdot 3 + 12) = 18$.

(e) The chromatic number $\chi(G)$ of G is 3 because 3 is the smallest number k for which $c_G(k) > 0$.

4. Prove that $z^4 - 4\,z^3 + 3\,z^2$ can not be the chromatic polynomial of any graph.

   Answer. If this is $c_G(z)$ for some graph G then $c_G(k) \geq 0$ for all $k \geq 1$ because $c_G(k)$ is the number is number of colorings of G with k colors. But $c_G(2) = 2^4 - 4\cdot 2^3 + 3\cdot 2^2 = -4 < 0$. Therefore, $z^4 - 4\,z^3 + 3\,z^2$ can not be the chromatic polynomial of any graph.

5. Indicate some important properties of the chromatic polynomial.