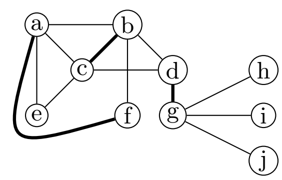
Seminar 7

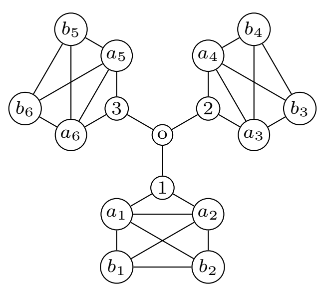
Matchings

You should be able to answer the following questions:

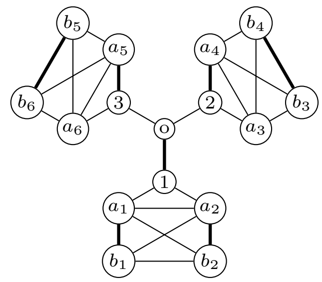
1. What is a matching of a simple undirected graph? What is a perfect matching?
2. Consider the matching marked with thick lines in the graph depicted below:

   Is this a maximum matching? Is this a maximal matching? Give reasons for your answers.
3. Does the following graph have a perfect matching? If yes, indicate a perfect matching. If no, explain why not.

   ANSWER: The main idea is to compute a maximum matching of this graph, and check if it is a perfect matching. For example, we can compute the following maximum matching M of the previous graph:

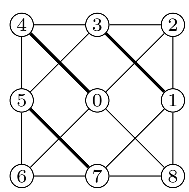


The only M-unsaturated nodes are a_3 and a_6, and there is no M-augmenting path between them $\Rightarrow$ M is a maximum matching. But M is not a perfect matching. If there were a perfect matching, it must be maximum. But we found a maximum matching which is not perfect. Thus, this graph has no perfect matching.

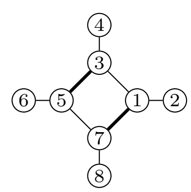
4. For each of the graphs depicted below, with the matchings M indicated with thick lines, indicate

   (a) an M-alternating path which is not an M-augmenting path.

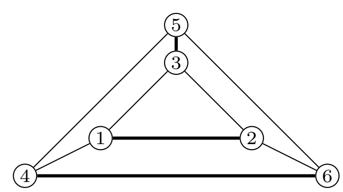
   (b) an M-augmenting path, if there is one; if there is one, use it to produce a larger matching.



G_1



G_2

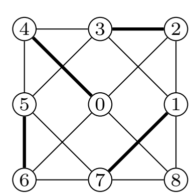


G_3

Answer:

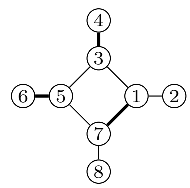
   (a) The following are M-alternating paths, but not M-augmenting paths:
   In G_1: (7,5,3,1,2)
   In G_2: (7,1,2)
   In G_3: (5,4,6,2,1,3)

   (b) The following are M-augmenting paths:
   In G_1: (2,3,1,7,5,6). By swapping the matchings along this path, we produce the larger matching



G_1

In G_2: (4,3,5,6). By swapping the matchings along this path, we produce the larger matching



G_2

G_3 has no M-augmenting paths; note that M is a perfect matching in G_3.

5. Indicate the statement of Hall Theorem for systems of distinct representatives.

6. For each family of sets indicated below, check if it fulfils the requirements of Hall Theorem. If yes, indicate a system of distinct representatives. If not, indicate the conditions which is not fulfilled, according to Hall Theorem.

   (a) $\{1,2,3\},\{2,3,4\},\{3,4,5\},\{4,5\},\{1,2,5\}$
   (b) $\{1,2,4\},\{2,4\},\{2,3\},\{1,2,3\}$
   (c) $\{1,2\},\{2,3\},\{1,2,3\},\{2,3,4\},\{1,3\},\{3,4\}$
   (d) $\{1,2,5\},\{1,5\},\{1,2\},\{2,5\}$

   Answer:

   (a) Fulfils the requirements of Hall Theorem. An SDR is
   $1 \in \{1,2,3\}, 3 \in \{2,3,4\}, 3 \in \{3,4,5\}, 4 \in \{4,5\}, 5 \in \{1,2,5\}$
   (b) Fulfils the requirements of Hall Theorem. An SDR is
   $2 \in \{1,2,4\}$, $4 \in \{2,4\}$, $3 \in \{2,3\}, 1 \in \{1,2,3\}$
   (c) Does not fulfil the requirements of Hall Theorem because we can find $t = 4$ sets whose union contains less than t elements:
   $\{1,2\} \cup \{2,3\} \cup \{1,2,3\} \cup \{1,3\} = \{1,2,3\}$ is a set with 3 elements.
   (d) Does not fulfil the requirements of Hall Theorem because there are $t = 4$ sets whose union contains less than t elements:
   $\{1,2,5\} \cup \{1,5\} \cup \{1,2\} \cup \{2,5\} = \{1,2,5\}$ is a set with 3 elements.

Weighted bipartite matchings

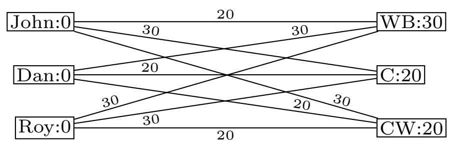
1. Use the Hungarian algorithm to solve the motivating problem from the lecture notes.

   Answer: Remember that

	wash bathroom (WB)	cook (C)	clean windows (CW)
John	20	30	30
Dan	30	20	20
Roy	30	30	20

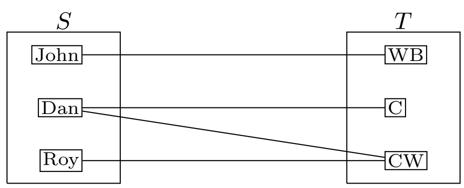
The weighted bipartite graph of the motivating problem is

G :



The initial costs assigned to the nodes are $c(\text{John}) = c(\text{Dan}) = c(\text{Roy}) = 0$, $c(\text{WB}) = \min\{20, 30, 30\} = 20$, $c(\text{C}) = \min\{30, 20, 30\} = 20$, $c(\text{CW}) = \min\{30, 20, 20\} = 20$. If we draw only the edges (x, y) for which $c(x) + c(y) = w(x, y)$, we obtain the graph

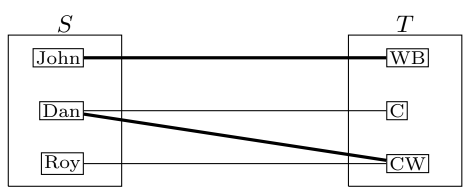
G_c :



Initially:
$M = \emptyset$
$R_S = S, R_T = T$
$Z = S \cup T$

- We have $Z \cap R_T = T \neq \emptyset$ (case 1), and can consider the M-alternating path (John, WB) $\Rightarrow$ we update $M = \{(\text{John}, \text{WB})\}$.
- Next step: $M = \{(\text{John}, \text{WB})\}$, $R_S = \{\text{Dan}, \text{Roy}\}$, $R_T = \{\text{C}, \text{CW}\}$, $Z = R_S \cup R_T$. We have $Z \cap R_T = R_T \neq \emptyset$ (case 1). This time, we can consider the M-alternating path (Dan, C) $\Rightarrow$ we update $M = \{(\text{John}, \text{WB}), (\text{Dan}, \text{CW})\}$.
- Next step: $M = \{(\text{John}, \text{WB}), (\text{Dan}, \text{CW})\}$, $R_S = \{\text{Roy}\}$, $R_T = \{\text{C}\}$, $Z = \{$Dan, Roy, C, CW$\}$:

  G_c :

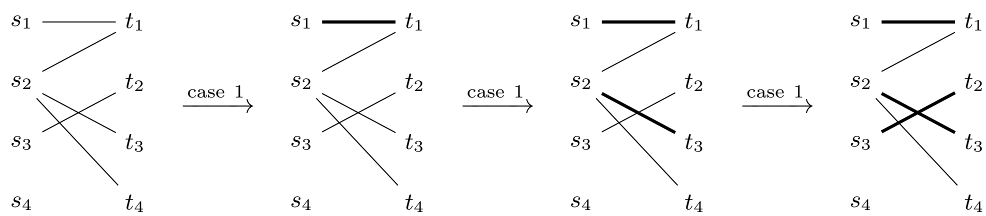
  We have $Z \cap R_T = C$ and we can consider the M-alternating path (Roy, CW, Dan, C) $\Rightarrow$ M becomes $\{(\text{John}, \text{WB}), (\text{Dan}, \text{C}), (\text{Roy}, \text{CW})\}$ and we stop.

2. Use the Hungarian algorithm to compute a minimal perfect matching in the complete bipartite graph G between $S = \{s_1, s_2, s_3, s_4\}$ and $T =$

$\{t_1, t_2, t_3, t_4\}$ where the weights are those from the following table:

	t_1	t_2	t_3	t_4
s_1	10	19	8	15
s_2	10	18	7	14
s_3	13	16	9	17
s_4	12	19	8	18

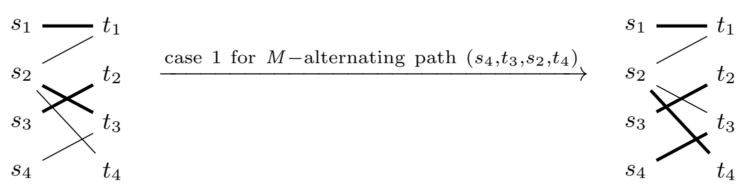
ANSWER: The initial cover of G is defined to be $c(s_1) = c(s_2) = c(s_3) = c(s_4) = 0$, $c(t_1) = 10$, $c(t_2) = 16$, $c(t_3) = 7$, $c(t_4) = 14$. Thus G_c and M evolve as follows:



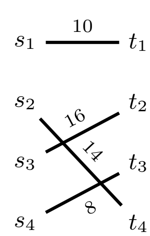
At this stage, we are in case 2, because $R_S = \{s_4\}$, $R_T = \{t_4\}$, $Z = R_S$ and $R_T \cap Z = \emptyset$. We compute

$$\begin{aligned}\delta &= \min\{w(x,y) - c(x) - c(y) \mid x \in Z \cap S, y \in T \setminus Z\} \\ &= \min\{w(s_4,t) - c(s_4) - c(t) \mid t \in T\} \\ &= \min\{12-10, 19-16, 8-7, 18-14\} = 1\end{aligned}$$

and modify c to be $c(s_1) = c(s_2) = c(s_3) = 0$, $c(s_4) = 1$, $c(t_1) = 10$, $c(t_2) = 16$, $c(t_3) = 7$, $c(t_4) = 14$. Now, G_c and M evolve as follows:

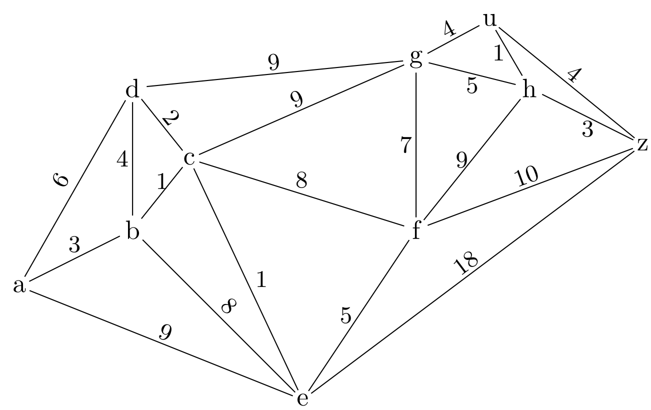


Thus, a minimum perfect matching between s and t is



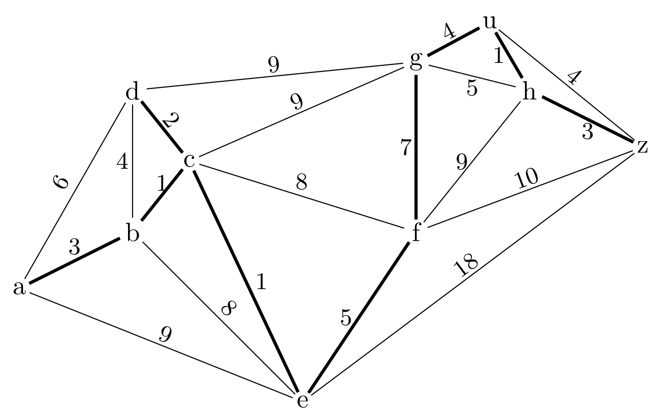
Minimum weight spanning trees

1. Apply Kruskal algorithm to determine and draw a minimum weight spanning tree of the connected graph depicted below



What is the value of the total weight of a minimum weight spanning tree?

ANSWER: An application of Kruskal algorithm could yield the following minimum weight spanning tree:



The minimum weight is $3 \cdot 1 + 1 \cdot 2 + 2 \cdot 3 + 1 \cdot 4 + 1 \cdot 5 + 1 \cdot 7 = \mathbf{27}$.

REMARK: The minimum weight spanning tree is not always unique. However, all minimum weight spanning trees have the same weight.

2. Compute a minimum weight spanning tree for the weighted graph