Seminar 6

Pólya's theory of counting – Lecture summary, exercises and homeworks –

1 Lecture summary

A **colouring** of a set of n elements $\{1, 2, \ldots, n\}$ is a map $c : \{1, 2, \ldots, n\} \to K$ where K is a finite set of colors.

- Every colouring c can be represented as a sequence $\langle k_1, k_2, \ldots, k_n\rangle$ where $k_i = c(i)$ for $1 \leq i \leq n$. If K contains m different colors, then the number of different colourings of $\{1, 2 \ldots, n\}$ is m^n.
- Given an n-permutation π and a set of colourings C, we define the map $\pi^* : C \to C$ by $\pi^*(c) = c'$ if $c'(i) = c(\pi(i))$.
  For example, if $c = \langle r, g, r, r\rangle$ and $\pi = (1, 2, 3, 4)$ then $\pi^*(c) = \langle g, r, r, r\rangle$.

Preliminary remarks:

- If G is a permutation group then the relation $\sim_G$ on colourings defined by

  $$c_1 \sim_G c_2 \text{ if there exists } \pi \in G \text{ such that } c_2 = \pi^*(c_1)$$

  is an equivalence relation. If $c_1 \sim_G c_2$, we say that c_1 and c_2 are indistinguishable w.r.t. G. For example, $\{\langle g, g, g, r\rangle, \langle g, g, r, g\rangle, \langle g, r, g, g\rangle, \langle r, g, g, g\rangle\}$ consists of colourings which are indistinguishable w.r.t. the permutation group C_4.
- Often, we are interested to count how many colourings can be distinguished w.r.t. a permutation group G. This number is the number of equivalence classes of $\sim_G$.
- **Burnside's lemma gives us a formula to compute the number of equivalence classes of $\sim_G$.**

Implicit assumptions from now on:

- C is the set of all possible colourings of n objects with m colours. C has m^n elements.
- G is a group of n-permutations. Typical examples are $G = D_n$ or $G = C_n$.

Useful auxiliary notions for $\pi \in G$ and $c \in C$:

Invariant set of π in C is $C_\pi = \{c \in C \mid \pi^*(c) = c\}$.

Stabilizer of c in G is $G_c = \{\pi \in G \mid \pi^*(c) = c\}$.

Equivalence class of c under the relation $\sim_G$ is $\overline{c} = \{\pi^*(c) \mid \pi \in G\}$.

Useful theoretical results:

- $|G_c| \cdot |\overline{c}| = |G|$ for all $c \in C$.
- Let N be the number of equivalence classes of $\sim_G$, that is, $N = |C/_{\sim_G}|$. According to Burnside's Lemma, we have

$$N = \frac{1}{|G|} \sum_{\pi \in G} |C_\pi|.$$

To find N, we must know how to compute the sizes $|C_\pi|$ of all invariant sets C_π.

- If C is the set of all possible colourings with m colours, and π consists of k cycles, then $|C_\pi| = m^k$.

Cycle index

The cycle index $P_G(x_1, x_2, \ldots, x_n)$ of a group G is a polynomial representation of the number and length of cycles in the permutations of G

$$P_G(x_1, x_2, \ldots, x_n) := \frac{\text{sum of monomials}}{|G|}$$

where $p \cdot x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_n^{\lambda_n}$ appears in the sum of monomials if and only if p is the number of permutations with type $[\lambda_1, \lambda_2, \ldots, \lambda_n]$ in G. This means that $p \cdot x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_n^{\lambda_n}$ if and only if p is the number of permutations in G whose cycle structure has λ_1 cycles of length 1, λ_2 cycles of length 2, ..., and λ_n cycles of length n. (Note that $1 \cdot \lambda_1 + 2 \cdot \lambda_2 + \ldots + n \cdot \lambda_n = n$.)

If π is a permutation with λ_i cycles of length i for $1 \leq i \leq n$ then $\pi^*(c) = c$ iff every cycle of π consists of elements with the same color

$\Rightarrow$ $|C_\pi| = m^{\lambda_1} m^{\lambda_2} \ldots m^{\lambda_n}$

$\Rightarrow$ By Burnside's Lemma, $N = P_G(m, m, \ldots, m)$.

Polya's enumeration formula

We consider a slightly more complicated coloruing problem:

Find the number $a_{(n_1, n_2, \ldots, n_m)}$ of colourings of n objects with m colours, if we assume that

- the colourings are undistinguishable w.r.t. the permutations of a permutation group G,
- we are constrained to use exactly n_1 times color y_1, exactly n_2 times color y_2, …, and exactly n_m times color y_m. Note that this implies that $n_1 + n_2 + \ldots + n_m = n$.

Polýa defined the **pattern inventory** polynomial

$$F_G(y_1, y_2, \ldots, y_m) = \sum_{n_1+n_2+\ldots+n_m} a_{(n_1,\ldots,n_m)} y_1^{n_1} y_2^{n_2} \ldots y_m^{n_m}$$

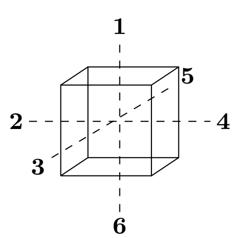
where $a_{(n_1,n_2,\ldots,n_m)}$ is the number defined before. Note that, if we manage to compute $F_G(y_1, y_2, \ldots, y_m)$, then we can read immediately the number of indistinguishable colorings for **all** possible distributions of colors $(n_1, n_2, \ldots, n_m)$ on n objects. Apparently, computing the polynomial F_G is much more difficult than solving the particular coloring problem mentioned at the beginning of this subsection. Pólya found a simple way to compute directly the polynomial F_G:

$$F_G(y_1, y_2, \ldots, y_m) = P_G\left(\sum_{i=1}^{m} y_i, \sum_{i=1}^{m} y_i^2, \ldots, \sum_{i=1}^{m} y_i^n\right).$$

2 Exercises and Homeworks

1. Determine the cycle index for the group of symmetries of the faces of a cube, and use this to determine the number of different six-sided dices that can be manufactured using 6 different labels for the faces of the dice. Assume ever colour is used exactly once.

   Answer: A cube has 6 faces which can be distinguished by numbering them from 1 to 6, as shown in the following figure. (The dashed lines are axes that cross the centers of opposite faces.)



   The group G of symmetries of the faces of the cube consists of the following 6-permutations:

   (a) The identity permutation (1)(2)(3)(4)(5)(6).

   (b) Multiple of 90° rotations around the dashed axes:

   Around axis 1-6: (1)(2, 3, 4, 5)(6), (1)(2, 4)(3, 5)(6), (1)(2, 5, 4, 3)(6)

Around axis 2-4: $(1,3,6,5)(2)(4)$, $(1,6)(2)(3,5)(2)$, $(1,5,6,3)(2)(4)$
Around axis 3-5: $(1,4,6,2)(3)(5)$, $(1,6)(2,4)(3)(5)$, $(1,2,6,4)(3)(5)$

(c) Multiple of 120° rotations around the axes through opposite corners of the cube (There are 3 such axes):

$(1,4,5)(2,3,6)$, $(1,5,4)(2,6,3)$
$(1,2,5)(3,6,4)$, $(1,5,2)(3,4,6)$
$(1,2,3)(4,5,6)$, $(1,3,2)(4,6,5)$
$(1,3,4)(2,6,5)$, $(1,4,3)(2,5,6)$

(d) 180° rotations around axes through the midpoints of opposite edges (There are 6 such axes):

$(1,2)(3,5)(4,6)$
$(1,5)(2,4)(3,6)$
$(1,4)(2,6)(3,6)$
$(1,3)(2,4)(5,6)$
$(1,6)(2,3)(4,5)$
$(1,6)(2,5)(3,4)$

In the end, we get a group with 24 elements, and the cycle index is

$$\begin{aligned} P_G(x_1,x_2,x_3,x_4,x_5,x_6) &= (x_1^6 + 3\,(2\,x_1^2 x_4 + x_1^2 x_2^2) + 8\,x_3^2 + 6\,x_2^3)/24 \\ &= (x_1^6 + 6\,x_1^2\,x_4 + 3\,x_1^2 x_2^2 + 8\,x_3^2 + 6\,x_2^3)/24 \end{aligned}$$

The pattern inventory polynomial for 6 colours y_1,y_2,y_3,y_4,y_5,y_6 is the following (one can use *Mathematica* to compute it):

$$\begin{aligned} F_G(y_1,y_2,y_3,y_4,y_5,y_6) &= P_G(\sum_{i=1}^{6} y_i, \sum_{i=1}^{6} y_i^2, \sum_{i=1}^{6} y_i^3, \sum_{i=1}^{6} y_i^4, \sum_{i=1}^{6} y_i^5, \sum_{i=1}^{6} y_i^6) \\ &= \text{a polynomial with 462 terms} \end{aligned}$$

The coefficient of $y_1 y_2 y_3 y_4 y_5 y_6$ in the pattern inventory polynomial is 30 $\Rightarrow$ there are 30 different cubes with faces coloured with y_1,y_2,y_3,y_4,y_5,y_6, where each colour is used exactly once. If, instead of y_1,y_2,y_3,y_4,y_5,y_6 we use numbers $1,2,3,4,5,6$, we can produce 30 different dices.

2. Use Pólya's enumeration formula to determine the number of six-sided dices that can be manufactured if each of three different labels must be placed on two of the faces.

   Answer: This problem can be solved by computing the pattern inventory for three kinds of labels. Let r,g,b be the three kinds of labels placed on the faces of the dice. Then the pattern inventory is the polynomial

   $$F_G(r,g,b) = P_G(r{+}g{+}b, r^2{+}g^2{+}b^2, r^3{+}g^3{+}b^3, r^4{+}g^4{+}b^4, r^5{+}g^5{+}b^5, r^6{+}g^6{+}b^6)$$

   where $P_G(x_1,x_2,x_3,x_4,x_5,x_6) = \frac{1}{24}\left(x_1^6 + 6\cdot x_1^2\,x_4 + 3\cdot x_1^2\,x_2^2 + 8\cdot x_3^2 + 6\cdot x_2^3\right)$

By replacing x_i with $r^i + g^i + b^i$ in $P_G(x_1, \ldots, x_6)$ for $1 \le i \le 6$ we obtain

$$\begin{aligned} F_G(r,g,b) = & b^6 + b^5 g + 2\,b^4 g^2 + 2\,b^3 g^3 + 2\,b^2 g^4 + b g^5 + g^6 + \\ & b^5 r + 2\,b^4 g r + 3\,b^3 g^2 r + 3\,b^2 g^3 r + 2\,b g^4 r + g^5 r + \\ & 2\,b^4 r^2 + 3\,b^3 g r^2 + 6\,b^2 g^2 r^2 + 3\,b g^3 r^2 + 2\,g^4 r^2 + \\ & 2\,b^3 r^3 + 3\,b^2 g r^3 + 3 b g^2 r^3 + 2 g^3 r^3 + 2\,b^2 r^4 + \\ & 2\,b g r^4 + 2\,g^2 r^4 + b r^5 + g r^5 + r^6. \end{aligned}$$

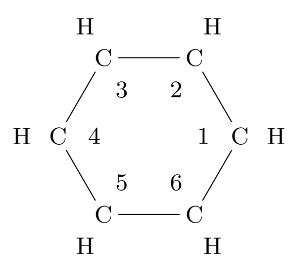
The coefficient of $r^2 g^2 b^2$ is 6. This means that there are 6 dices that can be manufactured using each of the three labels on two of the faces.

3. The hydrocarbon benzene has six carbon atoms arranged at the vertices of a regular hexagon, and six hydrogen atoms, with one bonded to each carbon atom. Two molecules are said to be *isomers* if they are composed of the same number and types of atoms, but have different structure.

   (a) Show that exactly three isomers (ortho-dichlorobenzene, meta-dichlorobenzene, and para-dichlorobenzene) may be constructed by replacing two of the hydrogen atoms of benzene with chlorine atoms.

   (b) How many isomers may be obtained by replacing two of the hydrogen atoms with chlorine atoms, and two others with bromine atoms?

   Answer: First, we must identify the group of symmetries of benzene. The structure of benzene is as shown below:



The symmetries of benzene are the elements of the dihedral group D_6. D_6 consists of the elements of C_6 (which are the multiple of 60° rotations around the center of symmetry) and its symmetries:

$$C_6 : \begin{cases} (1)(2)(3)(4)(5)(6), (1,2,3,4,5,6), \\ (1,3,5)(2,4,6), (1,4)(2,5)(3,6), \\ (1,5,3)(2,6,4), (1,6,5,4,3,2), \end{cases}$$

$$\text{symmetries of } C_6 : \begin{cases} (1,6)(2,5)(3,4), (1,2)(3,6)(4,5), (1,4)(2,3)(5,6), \\ (1)(2,6)(3,5)(4), (1,3)(2)(4,6)(5), (1,5)(2,4)(3)(6). \end{cases}$$

Te cycle index of D_6 is

$$P_{D_6}(x_1, x_2, \ldots, x_6) = \frac{1}{12}\left(x_1^6 + 2\,x_6 + 2\,x_3^2 + 4\,x_2^3 + 3\,x_1^2 x_2^2\right).$$

(a) Every isomer obtained by replacing a hydrogen with a chlorine can be interpreted as a coloring where we replace the color of the hydrogen (let's call it h) with the color of chlorine (let's call it c). Therefore, in order to count these isomers, we compute the pattern inventory polynomial for the coloring with 2 colors, h and c:

$$\begin{aligned} F_{D_6}(h,c) &= P_{D_6}\left(h+c, h^2+c^2, h^3+c^3, h^4+c^4, h^5+c^5, h^6+c^6\right) \\ &= c^6 + c^5h + 3\,c^4h^2 + 3\,c^3h^3 + 3\,c^2h^4 + ch^5 + h^6 \end{aligned}$$

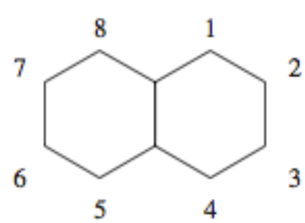
The coefficient of c^2h^4 is 3. This means that there we can produce 3 isomers by replacing 2 atoms of hydrogen with 2 atoms of chlorine.

(b) Every isomer obtained by replacing hydrogen with chlorine and/or bromine can be interpreted as a coloring of the nodes of benzene, which can be either h (for hydrogen), or c (for chlorine), or b (for bromine). Therefore, in order to count these isomers, we compute the pattern inventory polynomial for the coloring with 3 colors, h, c and b:

$$\begin{aligned} F_{D_6}(h,c,b) = P_{D_6}(\ & h+c+b, h^2+c^2+b^2, h^3+c^3+b^3, \\ & h^4+c^4+b^4, h^5+c^5+b^5, h^6+c^6+b^6) \\ = & b^6 + b^5c + 3\,b^4c^2 + 3\,b^3c^3 + 3\,b^2c^4 + bc^5 + c^6 + \\ & b^5h + 3\,b^4ch + 6\,b^3c^2h + 6\,b^2c^3h + 3\,bc^4h + c^5h + \\ & 3\,b^4h^2 + 6\,b^3ch^2 + 11\,b^2c^2h^2 + 6\,bc^3h^2 + 3\,c^4h^2 + \\ & 3\,b^3h^3 + 6\,b^2ch^3 + 6\,bc^2h^3 + 3\,c^3h^3 + 3\,b^2h^4 + \\ & 3\,bch^4 + 3\,c^2h^4 + bh^5 + ch^5 + h^6. \end{aligned}$$

The coefficient of $c^2h^2b^2$ is 11. This means that we can produce 11 isomers by replacing 2 atoms of hydrogen with chlorine atoms, and other 2 atoms of hydrogen with bromine atoms.

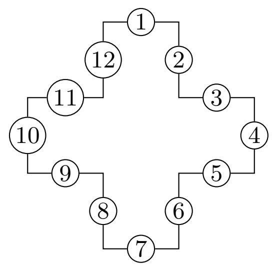
4. The hydrocarbon naphthalene has ten carbon atoms arranged in a double hexagon as in the figure below, and eight hydrogen atoms attached at each of the positions labeled 1 through 8.



   (a) Naphthol is obtained by replacing one of the hydrogen atoms of naphthalene with a hydroxyl group (OH). How many isomers of naphthol are there?

(b) Tetramethylnaphthalene is obtained by replacing four of the hydrogen atoms of naphthalene with methyl groups (CH_3). How many isomers of tetramethylnaphthalene are there?

5. Suppose a medical relief agency plans to design a symbol for their organization in the shape of a regular cross, as in the figure below. To symbolize the purpose of the organization and emphasize its international constituency, its board of directors decides that the cross should be white in color, with each of the twelve line segments outlining the cross colored red, green, blue, or yellow, with an equal number of lines of each colour. If we discount rotations and flips, how many different ways are there to design the symbol?



ANSWER: Let π be the cycle $(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12)$. The symmetries of this regular cross are:

- 4 rotations. These are:
  π^0 with type $[12, 0, \ldots, 0]$;
  π^3 and π^9 with type $[0, 0, 0, 1, 0, \ldots, 0]$; and
  π^6 with type $[0, 6, 0, \ldots, 0]$.
- 2 flips around horizontal and vertical axes. These are:
  $(1)(7)(2, 12)(3, 11)(4, 10)(5, 9)(6, 8)$ with type $[2, 5, 0, \ldots, 0]$, and
  $(4)(10)(3, 5)(2, 6)(1, 7)(8, 12)(9, 11)$ with type $[2, 5, 0, \ldots, 0]$.
- 2 flips around diagonal axes. These are:
  $(8, 9)(7, 10)(6, 11)(5, 12)(1, 4)(2, 3)$ with type $[0, 6, 0, \ldots, 0]$, and
  $(11, 12)(1, 10)(2, 9)(3, 8)(4, 7)(5, 6)$ with type $[0, 6, 0, \ldots, 0]$.

$\Rightarrow$ the cycle index polynomial is

$$P_G(x_1, x_2, \ldots, x_{12}) = (x_1^{12} + 2\,x_4^3 + 2\,x_1^2 x_2^5 + 3\,x_2^6)/8$$

For a colouring of the 12 edges with the four colours r, g, b, y, we have the pattern inventory polynomial

$$\begin{aligned} F_G(r, g, b, y) =& \big((r + g + b + y)^{12} + 2\,(r^4 + g^4 + b^4 + y^4)^3 + \\ & 2\,(r + g + b + y)^2 (r^2 + g^2 + b^2 + y^2)^5 + \\ & 3\,(r^2 + g^2 + b^2 + y^2)^6\big)/8 \\ =& \text{polynomial with 455 terms} \end{aligned}$$

The number of colourings of this regular cross with an equal number of lines of each colour is the coefficient of $r^3g^3b^3y^3$ in the pattern inventory. It turns out that this number is 46200.

Exercises related to Stirling cycle numbers

1. Use the table with the triangle of Stirling cycle numbers from the Lecture Notes to compute $\left[{9 \atop k}\right]$ for each k.
   ANSWER: Use the facts that

   ▷ $\left[{n \atop 0}\right] = 0$ and $\left[{n \atop n}\right] = 1$ for all $n > 0$, and

   ▷ the recursive formula $\left[{n \atop k}\right] = (n-1)\left[{n-1 \atop k}\right] + \left[{n-1 \atop k-1}\right]$

   to compute the next row of the triangle of Stirling cycle numbers.

2. Use a combinatorial argument to determine a simple formula for $\left[{n \atop n-2}\right]$.

   ANSWER: To seat n people at $n-2$ round tables we distinguish 2 disjoint possibilities:

   (a) We seat 3 people at a round table ⇒ all the remaining round tables are occupied by only one person.
   To count the possibilities in this case we proceed as follows. First, we must choose the 3 people to be seated together at a round table. There are $\binom{n}{3}$ such choices. Next, we must arrange them at the round table. Seating 3 people at a round table can be done in 2 ways. Thus, in this case there are $2 \cdot \binom{n}{3} = n(n-1)(n-2)/3$ possibilities.

   (b) There are 2 round tables occupied by 2 people; all others are occupied by only one person. In this case, the number of possibilities is $\binom{n}{2}\binom{n-2}{2} = n(n-1)(n-2)(n-3)/4$.

   By the Rule of Sum

$$\left[{n \atop n-2}\right] = \frac{n(n-1)(n-2)}{3} + \frac{n(n-1)(n-2)(n-3)}{4} = \frac{n(n-1)(n-2)(3\,n-5)}{12}$$

Exercises related to Stirling set numbers

1. Use the table with the triangle of Stirling set numbers from the Lecture Notes to compute $\left\{{9 \atop k}\right\}$ and $\left\{{10 \atop k}\right\}$ for each k.

   ANSWER: Use the facts that

   ▷ $\left\{{n \atop 0}\right\} = 0$ and $\left\{{n \atop n}\right\} = 1$ for all $n > 0$, and

   ▷ the recursive formula $\left\{{n \atop k}\right\} = k \cdot \left\{{n-1 \atop k}\right\} + \left\{{n-1 \atop k-1}\right\}$

   to compute the next two rows of the triangle of Stirling set numbers.

2. How many different fifty-character sequences use every character of the 26-letter alphabet at least once?

   Answer: Let A_i be the set of positions in the 50-character sequence where the i-th character from the alphabet occurs, E.g., if the sequence is

$$abcdefghijklmnopqrstuvwxyzaaaaaaaaaaaaaaaaaaaaaaaa$$

   then $A_1 = \{1, 27, 28, 29, \dots, 48, 49, 50\}$ and $A_i = \{i\}$ for $2 \le i \le 26$. Every fifty-character sequence that uses every character of the 26-letter alphabet at least once is uniquely determined by a partition of the set $\{1, 2, \dots, 50\}$ into 26 nonempty groups $A_1, \dots, A_{26}$, and vice-versa. Thus, the required number is $\left\{ {50 \atop 26} \right\}$.

3. Use combinatorial arguments to determine simple formulas for $\left\{ {n \atop 2} \right\}$ and $\left\{ {n \atop n-2} \right\}$ when $n \ge 2$.

   Answer: Let $N = \{1, 2, \dots, n\}$.

   $\left\{ {n \atop 2} \right\}$ is the number of non-empty partitions $\{A, N - A\}$ of N. This number is $M/2$, where M is the number of subsets of N except $\emptyset$ and N. We know that N has 2^n subsets. Thus $M = 2^n - 2$ and $\left\{ {n \atop 2} \right\} = (2^n - 2)/2 = 2^{n-1} - 1$.

   To count $\left\{ {n \atop n-2} \right\}$, we observe that the splitting of n elements in $n-2$ disjoint and nonempty groups can happen in only 2 ways:

   (a) One group has 3 elements, and all other groups have exactly 1 element. This can happen in $\binom{n}{3}$ ways.

   (b) There are 2 groups with 2 elements, and all other groups have exactly one element. This can happen in $\binom{n}{2} \cdot \binom{n-2}{2}$ ways.

   By the Rule of Sum

$$\left\{ {n \atop n-2} \right\} = \binom{n}{3} + \binom{n}{2} \cdot \binom{n-2}{2}.$$

4. Let $r_{n,k}$ denote the number of ways to divide n people into k groups, with at least *two* people in every group.

   Use a combinatorial argument to show that

$$r_{n,k} = k\, r_{n-1,k} + (n-1)\, r_{n-2,k-1}$$

   for $n \ge 1$.

   Answer: There are two ways to construct such a division of n people onto k groups. We distinguish two disjoint cases:

   - The first $n-1$ people are divided into k groups of at least 2 people each. There are $r_{n-1,k}$ such possible divisions. Person n can join any such group. By the Rule of Product, this case contributes with $k \cdot r_{n-1,k}$ possibilities.

- The first $n-1$ people are divided into k groups where 1 group consists of only one person, say m, and all other groups contain at least 2 people. Then n must join m to form a group of 2 people. In this case, the number of possibilities is the number of ways to divide the set $\{1, 2, \ldots, n-1\} \setminus \{m\}$ into $k-1$ groups with at least 2 people in each group. This number is $r_{n-2,k-1}$ for every choice of m from $\{1, 2, \ldots, n-1\}$. By the Rule of Product, this case contributes with $(n-1) \cdot r_{n-2,k-1}$ possibilities.

By the Rule of Sum, we learn that $r_{n,k} = k \cdot r_{n-1,k} + (n-1) \cdot r_{n-2,k-1}$.

Eulerian numbers (Exercise)

The Eulerian number $\left\langle {n \atop k} \right\rangle$ is the number of n-permutations π for which there are exactly k numbers i between 1 and $n-1$ such that $\pi(i) < \pi(i+1)$.

Give combinatorial proofs for the following relations:

1. $\left\langle {n \atop 0} \right\rangle = 0$ for all $n \geq 1$.

   ANSWER: The only n-permutation for which there is no i such that $\pi(i) < \pi(1+1)$ is $\langle n, n-1, \ldots, 2, 1\rangle$. Thus $\left\langle {n \atop 0} \right\rangle = 0$ for all $n \geq 1$.

2. $\left\langle {n \atop n-1} \right\rangle = 1$ for all $n \geq 1$.

   ANSWER: The only possibility to have $n-1$ numbers i between 1 and $n-1$ such that $\pi(i) < \pi(i+1)$ is when $\pi = \langle 1, 2, \ldots, n-1, n\rangle$. Thus $\left\langle {n \atop n-1} \right\rangle = 1$ for all $n \geq 1$.

3. $\left\langle {n \atop k} \right\rangle = (k+1)\left\langle {n-1 \atop k} \right\rangle + (n-k)\left\langle {n-1 \atop k-1} \right\rangle$ for all $n \geq 2$ and $1 \leq k < n$.

   ANSWER: Let $E_{n,k}$ be the set of n-permutations π for which there are exactly k numbers i between 1 and $n-1$ such that $\pi(i) < \pi(i+1)$. Every permutation of $E_{n,k}$ can be created in one of the following two ways:

   (a) Choose an $(n-1)$-permutation $\pi \in E_{n-1,k}$ and insert n in it, such that the newly created n-permutation is in $E_{n,k}$. We note that this happens if and only if:

   - n is inserted in the first position in π, or
   - n is inserted between positions i and $i+1$ of π for which $\pi(i) < \pi(i+1)$.

   Since $E_{n-1,k}$ contains $\left\langle {n-1 \atop k} \right\rangle$ permutations, and the insertion of n in it can be done in $k+1$ ways, this case contributes with $(k+1)\left\langle {n-1 \atop k} \right\rangle$ permutations to the construction of permutations from $E_{n,k}$.

   (b) Choose an $(n-1)$-permutation π from $E_{n-1,k-1}$ and insert n in it, such that the newly created n-permutation is in $E_{n,k}$. We note that

   - the set $J = \{j \mid \pi(j) > \pi(j+1)\}$ has $n-2-(k-1) = n-k-1$ elements, because $\pi \in E_{n-1,k-1}$ implies the existence of $k-1$ elements i between 1 and $n-2$ such that $\pi(i) < \pi(i+1)$.

- The insertion of n into π produces an n-permutation from $E_{n,k}$ if and only if n is inserted at the end of π, or between $\pi(j)$ and $\pi(j+1)$ where $j \in J$.

Since $E_{n-1,k-1}$ has $\left\langle {n-1 \atop k-1} \right\rangle$ elements, and the insertion of n can be done in $n-k$ ways, we conclude (by the Rule of Product) that this case contributes with $(n-k)\left\langle {n-1 \atop k-1} \right\rangle$ permutations to the construction of permutations from $E_{n,k}$.

By the Rule of Sum, we conclude that the number of elements of $E_{n,k}$ is

$$\left\langle {n \atop k} \right\rangle = (k+1)\left\langle {n-1 \atop k} \right\rangle + (n-k)\left\langle {n-1 \atop k-1} \right\rangle.$$

How to use *Mathematica* in this seminar?

Mathematica is a state-of-the-art system for technical computing. In particular, it can be used to perform symbolic computations, such as:

- polynomial computations: multiplications, divisions, expansions,
- exact computation with integer values of arbitrary size,
- solving various kinds of systems of equations,
- computer graphics, animation, etc.

Here, we describe how to use *Mathematica* to solve exercises for this seminar.

Start a *Mathematica* session

Double-click the *Mathematica* icon. A notebook will open.

In *Mathematica*, a notebook is a window in which the user can interact with the system. It consists of a sequence of cells, which are displayed one after the other. There are many kinds of cells:

- Input cells, which contain programs and definitions written in the programming language of *Mathematica*. Input cells can be *evaluated*: When the user/programmer clicks the mouse on the content of an input cell and then presses `Shift+Enter` (or `Ctrl+Enter`, it depends on the operating system), *Mathematica* will evaluate the content of the input cell and display the result of evaluation in an output cell.
- Output cells, that display the results of some evaluations.
- Text cells, which contain text intended to be read, not evaluated. There are many kinds of text cells: title, section, subsection, plain text, etc.

Computation and analysis of pattern inventories

Let's consider the computation of the pattern inventory of the faces of the cube for colourings with colours red (r), green (g), blue (b), white (w), yellow (y), and pink (p). We saw that the cycle index of the group of symmetries of the faces of the cube is

$$P_G(x_1, x_2, x_3, x_4, x_5, x_6) = (x_1^6 + 6\,x_1^2\,x_4 + 3\,x_1^2 x_2^2 + 8\,x_3^2 + 6\,x_2^3)/24$$

In *Mathematica*, we can define this polynomial by writing and evaluating an input cell with the following content:

```
PG[x1_,x2_,x3_,x4_,x5_,x6_]:=
   (x1^6 + 6 x1^2 x4 + 3 x1^2 x2^2 + 8 x3^2 + 6 x2^3) / 24
```

This is a definition of the index polynomial `PG` as a function of the indeterminates `x1`, `x2`, `x3`, `x4`, `x5`, and `x6`. Note that we write

`expr^i`

for the expression `expr` raised to power `i`, and

`expr1 expr2`

for the product of the expressions `expr1` and `expr2`.

In order to find the inventory polynomial for the colours r, g, b, w, y, p, we must replace in `PG` every `x`i with $r^i + g^i + b^i + w^i + y^i + p^i$ for $1 \leq i \leq 6$, and perform all polynomial additions and multiplications until we obtain a sum of terms. This operation s called *expansion*, and *Mathematica* performs it if we evaluate the following input cell

```
FG = Expand[
       r+g+b+w+y+p, r^2+g^2+b^2+w^2+y^2+p^2,
       r^3+g^3+b^3+w^3+y^3+p^3, r^4+g^4+b^4+w^4+y^4+p^4,
       r^5+g^5+b^5+w^5+y^5+p^5, r^6+g^6+b^6+w^6+y^6+p^6]
```

This operation is hopeless to perform by hand: it is a polynomial with 462 terms! Every term is of the form $a\,r^{i_1} g^{i_2} b^{i_3} w^{i_4} y^{i_5} p^{i_6}$ where a represents the number of colourings of the cube if i_1 faces are red, i_2 are green, i_3 are blue, i_4 are white, i_5 are yellow, and i_6 are pink. For example, the number of different cubes made with 3 red faces, 2 green faces, and one blue face is the coefficient of $r^3 g^2 b$ in the polynomial `FG`. We can find this number in 2 ways:

- The "hard way", by trying to find the coefficient of $r^3 g^2 b$ while reading the polynomial `FG`.
- The direct way, by asking *Mathematica* to find that coefficient for us. In this case, we evaluate the input cell

  ```
  Coefficient[FG,r^3g^2 b]
  ```

Both ways we get answer is 3. Thus, there are 3 different cubes with 3 red faces, 2 green, and one red.

Combinatorics

Mathematica has built-in support for combinatorics. We mention here only the fact that:

- The expression $n!$ is evaluated to the factorial of n. For example `41!` is evaluated to `33452526613163807108170062053440751665152000000000`.
- `Binomal[n,m]` evaluates to $\binom{n}{m}$.
- `Table[expr,i,m,n]` creates the expression

  $$\{expr_n, expr_{m+1} \ldots, expr_{n-1}, \ldots, expr_n\}$$

  where $expr_j$ is the result of replacing i with a value j between m and n in $expr$.
  NOTE: *Mathematica* represents lists as sequences of elements between curly braces. Thus, from the point of view of *Mathematica*, $\{e_1, \ldots, e_n\}$ represents the list of elements $e_1, \ldots, e_n$ in this order.

For example, the evaluation of an input cell with the content

```
Table[Binomial[10,i],{i,0,10}]
```

yields the list $\{1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1\}$ of all binomial values $\binom{10}{i}$ for $0 \leq i \leq 10$. *Mathematica* also knows row to compute Stirling cycle numbers and Stirling set numbers. The built-in methods `StirlingS1`$[n, m]$ and `StirlingS2`$[n, m]$ are provided for this purpose.