Graph Theory and Combinatorics

Seminar 3

1. Arrange the following permutations of $\{1,2,3,4,5\}$ in increasing lexicographic order: $\langle 4,3,1,2,5\rangle, \langle 2,1,5,3,4\rangle, \langle 3,4,5,2,1\rangle, \langle 3,4,1,5,2\rangle, \langle 4,3,1,5,2\rangle$.

2. What is the permutation that follows after $\langle 4,5,3,2,1\rangle$ in lexicographic order?

3. What is the permutation that follows after $\langle 4,2,5,3,1\rangle$ in lexicographic order?

4. What is the rank of the 3-permutation with repetition $\langle 2,2,5\rangle$ of the set $\{1,2,3,4,5\}$ in the lexicographic ordering?

   Answer: $1,2,3,4,5$ have ranks $0,1,2,3,4$ in the set $\{1,2,3,4,5\}$ (which has 5 elements), thus the rank of the 3-permutation with repetition $\langle 2,2,5\rangle$ is $114_{(5)} = 1\cdot 5^2 + 1\cdot 5 + 4 = 25+5+4 = 34$.

5. Compute the 6-permutation with repetition of the set $\{1,2\}$ with rank 17 in the lexicographic ordering.

   Answer: The set $A = \{1,2\}$ has 2 elements, thus we shall compute the representation in base 2 of 17, using 6 digits:

   $$17 = 16 + 1 = 2^4 + 1 = 010001_{(2)}$$

   and the 6-permutation with repetition of $\{1,2\}$ corresponding to the binary number $010001_{(2)}$ is $\langle 121112\rangle$.

6. Compute the 5-permutation with repetition of the set $\{1,2,3\}$ with rank 17 in the lexicographic ordering.

7. Which is the fourth subset of $\{1,2,3,4\}$ enumerated with the standard reflected Grey code?

8. Which is the subset of $\{1,2,3,4\}$ that follows after $\{1,3\}$ in the enumeration order via binary representations?

   Answer: $4 > 3 > 2 > 1$ and the binary representation of $\{1,3\}$ is 0101. The next binary encoding is 0110, which corresponds to the subset $\{2,3\}$.

9. Which is the 3-combination of $\{1,2,3,4,5\}$ that follows after $\{1,3,4\}$ in the enumeration order via binary representations? ANSWER:

   $5 > 4 > 3 > 2 > 1$ and

3-combinaton	binary representation 54321
$\{1,3,4\}$	01101

   The next 5-bit number with 3 occurrences of 1, after 01101, is 01110.
   The 3-combination for 01110 is $\{2,3,4\}$.

10. What is the rank of the permutation $\langle 1,2,4,5,3\rangle$ in lexicographic order?

11. What is the rank of the permutation $\langle 2,4,1,3\rangle$ in lexicographic order?

12. Write down the canonical cyclic structures of the following permutations:

    (a) $\langle 1,6,3,2,5,7,4,9,8\rangle$

    (b) $\langle 6,5,8,1,2,4,7,3,10,9\rangle$

    (c) $\langle 6,1,2,4,5,3,8,7\rangle$

13. Write down the permutations with the following cyclic structures:

    (a) $(4,7,3)(8,5,1,2)(10,6,9)$

    (b) $(6)(1,2,3,4,5)$

    (c) $(2,4,6,8)(1,5,3,7)$

14. Write down the canonical cyclic structures of the cyclic structures from the previous exercise.

15. Write down the types of the following permutations:

a) $\langle 1,2,3,4,5,6,7,8\rangle$	b) $\langle 8,1,2,3,4,5,6,7\rangle$
c) $\langle 4,5,6,7,1,2,3\rangle$	d) $\langle 3,4,5,6,7,1,2\rangle$
e) $\langle 1,2,3,4,6,5,8,7\rangle$	f) $\langle 7,8,5,6,4,3,2,1\rangle$

16. Which of the following lists is a valid permutation type:

(a) $[1,1,0,1]$	(b) $[1,1,0,0]$
(c) $[1,0,1,0,0]$	(d) $[0,0,0,0,1]$
(e) $[0,1,0,1,0,0]$	(f) $[1,0,1,0,0,0]$

17. Write down all plausible types of the permutations of the set $\{1, 2, 3, 4\}$. [SUGGESTION: Write down all integer partitions of 4, and use the relationship between integer partitions and permutation types described in Lecture 4.]

18. Write down all plausible types of the permutations of the set $\{1, 2, 3\}$.

19. How many permutations have the same type as the permutation $\langle 3, 2, 1, 5, 4\rangle$?

20. Which of the following formulas is a solution of the recurrence relation $a_n = 8\,a_{n-1} - 16\,a_{n-2}$:

a) $a_n = 0$?	b) $a_n = 1$?
c) $a_n = 2^n$?	d) $a_n = 4^n$?
e) $a_n = n \cdot 4^n$?	f) $a_n = 2 \cdot 4^n + 3 \cdot n \cdot 4^n$?
g) $a_n = (-4)^n$?	h) $a_n = n^2 \cdot 4^n$?

21. Solve these recurrence relations together with the initial conditions given.

    (a) $a_n = 2\,a_{n-1}$ for $n \geq 1$, $a_0 = 3$.

    (b) $a_n = 5\,a_{n-1} - 6\,a_{n-2}$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$.

    (c) $a_n = 4\,a_{n-1} - 4\,a_{n-2}$, $a_0 = 6$, $a_1 = 8$.

    (d) $a_n = 4\,a_{n-2}$, $a_0 = 0$, $a_1 = 4$.

    (e) $a_n = 7\,a_{n-1} - 10\,a_{n-2}$ for $n \geq 2$, $a_0 = 2$, $a_1 = 1$.

22. Find the solution to $a_n = 2\,a_{n-1} + 5\,a_{n-2} - 6\,a_{n-3}$ with $a_0 = 7$, $a_1 = -4$, and $a_2 = 8$.

    ANSWER: The characteristic equation of this relation is $r^3 - 2\,r^2 - 5\,r + 6 = 0$. As a first guess, we will check if some divisor of the free term 8 is root of this equation. It turns out that $r_1 = 1$ is solution, thus $r^3 - 2\,r^2 - 5\,r + 6$ is divisible with $r - 1$. If we divide $r^3 - 2\,r^2 - 5\,r + 6$ with $r - 1$, we get $r^3 - 2\,r^2 - 5\,r + 6 = (r^2 - r - 6)(r - 1) = (r + 2)(r - 3)(r - 1)$, thus the characteristic equation has three distinct roots: $r_1 = 1$, $r_2 = -2$, $r_3 = 3$. It follows that

$$a_n = a \cdot r_1^n + b \cdot r_2^n + c \cdot r_3^n = a + b \cdot (-2)^n + c \cdot 3^n$$

    From $a_0 = 7$, $a_1 = -4$, and $a_2 = 8$ we obtain

$$\left\{\begin{array}{rl} a + b + c & = 7 \\ a - 2\,b + 3\,c & = -4 \\ a + 4\,b + 9\,c & = 8 \end{array}\right. \Rightarrow \left\{\begin{array}{ll} a & = 5 \\ b & = 3 \\ c & = -1 \end{array}\right. \Rightarrow a_n = 5 + 3 \cdot (-2)^n - 3^n.$$

23. Determine the values of the constants a and b such that $a_n = a\,n + b$ is a solution of the recurrence relation $a_n = 2\,a_{n-1} + n + 5$.

ANSWER: If we replace a_n with $a\,n + b$ in this recurrence relation, we obtain

$$a\,n + b = 2\,(a\,(n-1) + b) + n + 5 = (2\,a + 1)\,n + 2\,b - 2\,a + 5$$

for all $n \in \mathbb{N}$, which implies

$$\begin{cases} a &= 2\,a + 1 \\ b &= 2\,b - 2\,a + 5 \end{cases} \Rightarrow \begin{cases} a &= -1 \\ b &= -7 \end{cases}$$

24. What is the general form of a solution of the linear nonhomogeneous recursive relation $a_n = 6\,a_{n-1} - 12\,a_{n-2} + 8\,a_{n-3} + F(n)$, if

    (a) $F(n) = n^2$?
    (b) $F(n) = n\,2^n$?
    (c) $F(n) = n^2\,2^n$?
    (d) $F(n) = 2^n$?
    (e) $F(n) = (-2)^n$?

25. Find all solutions of the recurrence relation $a_n = 2\,a_{n-1} + 2\,n^2$.

26. Let

$$a_n = \sum_{k=1}^{n} \frac{k(k+1)}{2} \quad \text{for all } n \geq 1.$$

Show that $\{a_n\}$ satisfies the linear nonhomogeneous recurrence relation

$$a_n = a_{n-1} + \frac{n(n+1)}{2}$$

and the initial condition $a_1 = 1$. Then, solve this recurrence relation to determine a formula for a_n.

27. A computer system considers a string of decimal digits a valid codeword if it contains an even number of 0 digits. For instance, 12304056789 is valid, whereas 12098700567 is not. Let a_n be the number of valid n-digit codewords. Find a recurrence relation for a_n.

ANSWER: There are nine 1-digit codewords, namely 1,2,3,4,5,6,7,8,9. Thus $a_1 = 9$.

If $n > 1$, we distinguish 2 disjoint cases:

1. The codeword starts with a non-zero digit x, thus it is of the form xw where $1 \le x \le 9$ and w id a codeword with $n-1$ decimal digits. There are 9 choices for x, and a_{n-1} choices for w. By the rule of product, there are $9 \cdot a_{n-1}$ codewords in this case.
2. The codeword starts with 0, thus it is of the form $0w'$ where w' is a string with $n-1$ decimal digits which is not a codeword. There are 10^{n-1} strings with $n-1$ decimal digits. By the principle of inclusion and exclusion, there are $10^{n-1} - a_{n-1}$ possibilities to choose w'. Thus, there are $10^{n-1} - a_{n-1}$ codewords in this case.

By the sum rule, $a_n = 9\,a_{n-1} + 10^{n-1} - a_{n-1} = 10^{n-1} + 8\,a_{n-1}$.

28. Find a recurrence relation for C_n, the number of ways to parenthesize the product of $n+1$ numbers, $x_0 \cdot x_1 \cdot x_2 \cdot \ldots \cdot x_n$, to specify the order of multiplication. For example $C_3 = 5$ because there are five ways to parenthesize $x_0 \cdot x_1 \cdot x_2 \cdot x_3$ to determine the order of multiplication:

$$\begin{array}{lll} ((x_0 \cdot x_1) \cdot x_2) \cdot x_3 & (x_0 \cdot (x_1 \cdot x_2)) \cdot x_3 & (x_0 \cdot x_1) \cdot (x_2 \cdot x_3) \\ x_0 \cdot ((x_1 \cdot x_2) \cdot x_3) & x_0 \cdot (x_1 \cdot (x_2 \cdot x_3)) & \end{array}$$

Answer: Note that, however we insert parentheses in the product $x_0 \cdot x_1 \cdot x_2 \cdot \ldots \cdot x_n$, one '$\cdot$' operator remains outside all parentheses, namely, the operator for the final multiplication operation to be performed. For example, in $((x_0 \cdot x_1) \cdot x_2) \cdot x_3$ it is the final '$\cdot$', while in $(x_0 \cdot x_1) \cdot (x_2 \cdot x_3)$ it is the second '$\cdot$'. This final operator appears between two of the $n+1$ numbers, say between x_k and x_{k+1}.

By the product rule, there are $C_k \cdot C_{n-k-1}$ ways to insert parentheses to determine the order of the $n+1$ numbers to be multiplied when the final operator appears between x_k and x_{k+1}, because:

- There are C_k ways to parenthesize the product $x_0 \cdot x_1 \cdot \ldots \cdot x_k$, and
- There are C_{n-k-1} ways to parenthesize the product $x_{k+1} \cdot \ldots \cdot x_n$.

Because the final operator can appear between any of the $n+1$ numbers, it follows from the rule of sum that

$$C_n = \sum_{k=0}^{n-1} C_k \cdot C_{n-k-1}.$$

Note that the initial conditions are $C_0 = 1$ and $C_1 = 1$.
From now we can compute the other values of C_n recursively:

$C_2 = C_0 \cdot C_1 + C_1 \cdot C_0 = 1 + 1 = 2$
$C_3 = C_0 \cdot C_2 + C_1 \cdot C_1 + C_2 \cdot C_0 = 1 \cdot 2 + 1 \cdot 1 + 2 \cdot 1 = 5$
$C_4 = C_0 \cdot C_3 + C_1 \cdot C_2 + C_2 \cdot C_1 + C_3 \cdot C_0 = 5 + 2 + 2 + 5 = 14$
$\ldots$

29. A vending machine dispensing books of stamps accepts only bills of 1\$, 5\$, and 20\$. Let b_n be the number of ways to deposit n dollars in the vending machine, if the order in which the bills are inserted matters. Find a recursive formula for the computation of b_n.

    Answer: We distinguish three distinct cases:

    1. The first bill inserted in the vending machine is 1\$ $\Rightarrow$ there are $n-1$ dollars left to insert in the machine, and this can be done in a_{n-1} ways.
    2. The first bill inserted in the vending machine is 5\$ $\Rightarrow$ there are $n-5$ dollars left to insert in the machine, and this can be done in a_{n-5} ways.
    3. The first bill inserted in the vending machine is 20\$ $\Rightarrow$ there are $n-20$ dollars left to insert in the machine, and this can be done in a_{n-20} ways.

    By the sum rule, $a_n = a_{n-1} + a_{n-5} + a_{n-20}$.

30. Let a_n be the number of n-bit strings that contain 01. Find a recurrence relation for the computation of a_n.
    Answer: Let b_n be the number of all n-bit strings, and c_n be the number of all n-nit strings that do not contain 01. An n-bit string is of one of the following two kinds:

    (a) It contains 01. There are a_n n-bit strings of this kind.
    (b) It does not contain 01. There are c_n n-bit strings of this kind.

    By the rule of sum, the total number of n-bit strings is $b_n = a_n + c_n$. By the rule of product, $b_n = 2^n$. Thus $a_n = b_n - c_n = 2^n - c_n$.

    An n-bit string which does not contain 01 is of the form

    $$\underbrace{1\ldots1}_{k \text{ times}}\ \underbrace{0\ldots0}_{n-k \text{ times}}$$

    where $0 \leq k \leq n$. There are $n+1$ possible values for k, thus $c_n = n + 1$.

    We conclude that $a_n = 2^n - (n+1)$.