Polya theory of counting

Problem 1

> How many distinct dices can be manufactured if one uses 3 different colours to color the faces of the dice and each colour is used to color two faces of the dice?

Answer

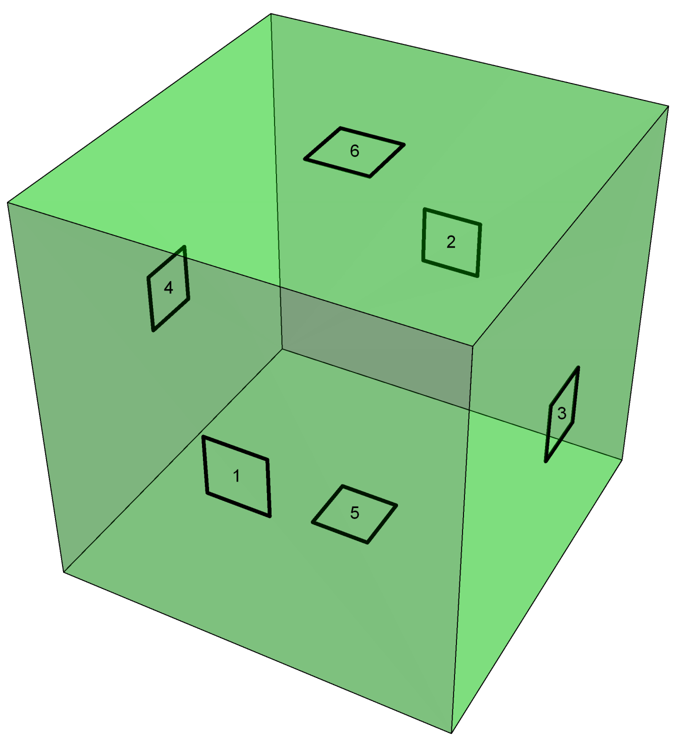
Consider the following colours: red (r), green (g) and blue (b) that are used to color the faces of the dice

We have to calculate the number of colourings of the dice in which there are 2 red faces, 2 green faces and 2 blue faces. This number can be obtained by computing the pattern inventory polynomial for 3 colours: F_G(r,g,b) of the group G of symmetries of the faces of the dice.

- First we determine the group of symmetries of the faces of the dice.

We use Mathematica for drawing the dice with faces numbered from 1 to 6

In[1]:=
```
Graphics3D[{Green, Opacity[.3], Cuboid[{-1, -1, -1}, {1, 1, 1}], Black, Opacity[1],
  Text["1", {0, -1, 0}], Text["2", {0, 1, 0}], Text["4", {-1, 0, 0}],
  Text["3", {1, 0, 0}], Text["5", {0, 0, -1}], Text["6", {0, 0, 1}], Thick,
  Line[
   {{-.15, -.15, 1}, {-.15, .15, 1}, {.15, .15, 1}, {.15, -.15, 1}, {-.15, -.15, 1}}],
  Line[{{-.15, -.15, -1}, {-.15, .15, -1}, {.15, .15, -1},
    {.15, -.15, -1}, {-.15, -.15, -1}}],
  Line[{{-.15, -1, -.15}, {-.15, -1, .15}, {.15, -1, .15},
    {.15, -1, -.15}, {-.15, -1, -.15}}],
  Line[{{-.15, 1, -.15}, {-.15, 1, .15}, {.15, 1, .15},
    {.15, 1, -.15}, {-.15, 1, -.15}}],
  Line[{{1, -.15, -.15}, {1, -.15, .15}, {1, .15, .15},
    {1, .15, -.15}, {1, -.15, -.15}}],
  Line[{{-1, -.15, -.15}, {-1, -.15, .15}, {-1, .15, .15},
    {-1, .15, -.15}, {-1, -.15, -.15}}]}, Boxed → False]
```

Out[1]=


(Reflection symmetries and Rotational symmetries) https://www.youtube.com/watch?v=xVYa9orJv08
(Reflections, Rotations, Translations) https://www.youtube.com/watch?v=VJTxv-tRKj0
(Rotating a cube-- indentifying the axes) https://www.youtube.com/watch?v=7GkuqcoGCU4
Step by step:
https://www.youtube.com/watch?v=-PYDcHKPMKk
https://www.youtube.com/watch?v=TggbcOrALMQ
https://www.youtube.com/watch?v=X3eOGQGntEs
Other: https://www.youtube.com/watch?v=vbfjA4AFedM

The group G of symmetries of the faces of the dice consists of the following 6-permutations:

- The identity permutation (1)(2)(3)(4)(5)(6) --> the monomial x_1^6
- Multiple rotations of 90^0 or 270^0 around the dashed axes (the axes through the middle of the opposite faces). E.g., the permutations for these rotations around axis 1-2 are: (1)(2)(3,6,4,5) and (1)(2)(3,5,4,6), and each one of them contributes with monomial $x_1^2\,x_4$. There are 3 axis through the middle of the opposite faces ⇒ the sum of the monomials for rotations of 90^0 or 270^0 is $6\,x_1^2\,x_4$.
- Multiple rotations of 180^0 around the dashed axes. E.g., the permutation for this rotation around axis 1-2 is (1)(2)(3,4)(5,6), and contributes with the monomial $x_1^2\,x_2^2$. There are 3 axis through the middle of the opposite faces ⇒ the sum of the monomials for rotations of 180^0 is $3\,x_1^2\,x_2^2$.
- Multiple rotations of 120^0or 240^0 through opposite corners of the dice: (1,6,4)(3,2,5), (1,4,6)(2,3,5), … There are 8 permutations for this kind of rotations, and the sum of monomials for these is $8\,x_3^2$.
- Multiple rotations of 180^0 through the midpoints of opposite edges: (1,3)(2,4)(5,6),… There are 6 such axis, and the sum of monomials for these permutations is $6\,x_2^3$.

In the end, we get a group G with 24 permutations, and the cycle index of G is

In[2]:=
```
PG[x1_, x2_, x3_, x4_, x5_, x6_] := 1/24 (x1^6 + 6 x1^2 x4 + 3 x1^2 x2^2 + 8 x3^2 + 6 x2^3)
```

We apply the Polya's counting formula to compute the pattern inventory polynomial for three colours r,g,b:

In[3]:=
```
FG = PG[r + g + b, r^2 + g^2 + b^2, r^3 + g^3 + b^3, r^4 + g^4 + b^4, r^5 + g^5 + b^5, r^6 + g^6 + b^6]
```

Out[3]=

$$\frac{1}{24}\left((b+g+r)^6+3\,(b+g+r)^2\left(b^2+g^2+r^2\right)^2+6\left(b^2+g^2+r^2\right)^3+8\left(b^3+g^3+r^3\right)^2+6\,(b+g+r)^2\left(b^4+g^4+r^4\right)\right)$$

In[4]:=
```
FG = Expand[PG[r + g + b, r^2 + g^2 + b^2, r^3 + g^3 + b^3, r^4 + g^4 + b^4, r^5 + g^5 + b^5, r^6 + g^6 + b^6]]
```

Out[4]=

$$b^6+b^5\,g+2\,b^4\,g^2+2\,b^3\,g^3+2\,b^2\,g^4+b\,g^5+g^6+b^5\,r+2\,b^4\,g\,r+3\,b^3\,g^2\,r+3\,b^2\,g^3\,r+2\,b\,g^4\,r+g^5\,r+2\,b^4\,r^2+3\,b^3\,g\,r^2+6\,b^2\,g^2\,r^2+3\,b\,g^3\,r^2+2\,g^4\,r^2+2\,b^3\,r^3+3\,b^2\,g\,r^3+3\,b\,g^2\,r^3+2\,g^3\,r^3+2\,b^2\,r^4+2\,b\,g\,r^4+2\,g^2\,r^4+b\,r^5+g\,r^5+r^6$$

or

In[5]:=
```
Expand[FG]
```

Out[5]=

$$b^6+b^5\,g+2\,b^4\,g^2+2\,b^3\,g^3+2\,b^2\,g^4+b\,g^5+g^6+b^5\,r+2\,b^4\,g\,r+3\,b^3\,g^2\,r+3\,b^2\,g^3\,r+2\,b\,g^4\,r+g^5\,r+2\,b^4\,r^2+3\,b^3\,g\,r^2+6\,b^2\,g^2\,r^2+3\,b\,g^3\,r^2+2\,g^4\,r^2+2\,b^3\,r^3+3\,b^2\,g\,r^3+3\,b\,g^2\,r^3+2\,g^3\,r^3+2\,b^2\,r^4+2\,b\,g\,r^4+2\,g^2\,r^4+b\,r^5+g\,r^5+r^6$$

In[7]:=
```
Length[FG]
```

Out[7]= 28

⇒ The number of distinct colourings of the dice with 2 red faces, 2 green faces and 2 blue faces is the coefficient of the monomial $r^2\,g^2\,b^2$, namely 6.

In[8]:= **Coefficient[FG, r^2 g^2 b^2]**

Out[8]= 6

Problem 2

Use Polya’s enumeration formula to determine the number of six-sided dices that can be manufactures if we use 6 colours. Assume that each colour is used exactly once.

Answer

Consider the following colours: y1, y2, y3, y4, y5, y6 that are used to color the faces of the dice
We have to calculate the number of colourings of the dice where each colour is used once. This number can be obtained by computing the pattern inventory polynomial for 6 colours: F_G(y1,y2,y3,y4,y5,y6) of the group G of symmetries of the faces of the dice.

The group G with 24 permutations, and the cycle index of G is the one obtained at Problem 1

In[9]:= $$P_G[x1\_, x2\_, x3\_, x4\_, x5\_, x6\_] := \frac{1}{24}\left(x1^6 + 6\, x1^2\, x4 + 3\, x1^2\, x2^2 + 8\, x3^2 + 6\, x2^3\right)$$

For a colouring of 6 sides with six colours y1, y2, y3, y4, y5, y6 we have the pattern inventory polynomial:

In[10]:= $$FG2 = P_G\left[y1 + y2 + y3 + y4 + y5 + y6,\ y1^2 + y2^2 + y3^2 + y4^2 + y5^2 + y6^2,\ y1^3 + y2^3 + y3^3 + y4^3 + y5^3 + y6^3,\ y1^4 + y2^4 + y3^4 + y4^4 + y5^4 + y6^4,\ y1^5 + y2^5 + y3^5 + y4^5 + y5^5 + y6^5,\ y1^6 + y2^6 + y3^6 + y4^6 + y5^6 + y6^6\right]$$

Out[10]= $$\frac{1}{24}\Big((y1 + y2 + y3 + y4 + y5 + y6)^6 + 3\,(y1 + y2 + y3 + y4 + y5 + y6)^2\left(y1^2 + y2^2 + y3^2 + y4^2 + y5^2 + y6^2\right)^2 + 6\left(y1^2 + y2^2 + y3^2 + y4^2 + y5^2 + y6^2\right)^3 + 8\left(y1^3 + y2^3 + y3^3 + y4^3 + y5^3 + y6^3\right)^2 + 6\,(y1 + y2 + y3 + y4 + y5 + y6)^2\left(y1^4 + y2^4 + y3^4 + y4^4 + y5^4 + y6^4\right)\Big)$$

In[12]:= **Expand[FG2]**

Out[12]= $$y1^6 + y1^5\, y2 + 2\, y1^4\, y2^2 + 2\, y1^3\, y2^3 + 2\, y1^2\, y2^4 + y1\, y2^5 + y2^6 + y1^5\, y3 + 2\, y1^4\, y2\, y3 + 3\, y1^3\, y2^2\, y3 + 3\, y1^2\, y2^3\, y3 + 2\, y1\, y2^4\, y3 + y2^5\, y3 + 2\, y1^4\, y3^2 + 3\, y1^3\, y2\, y3^2 + 6\, y1^2\, y2^2\, y3^2 + 3\, y1\, y2^3\, y3^2 + 2\, y2^4\, y3^2 + 2\, y1^3\, y3^3 + 3\, y1^2\, y2\, y3^3 + 3\, y1\, y2^2\, y3^3 + 2\, y2^3\, y3^3 + 2\, y1^2\, y3^4 + 2\, y1\, y2\, y3^4 + 2\, y2^2\, y3^4 + y1\, y3^5 + y2\, y3^5 + y3^6 + y1^5\, y4 + 2\, y1^4\, y2\, y4 + 3\, y1^3\, y2^2\, y4 + 3\, y1^2\, y2^3\, y4 + 2\, y1\, y2^4\, y4 + y2^5\, y4 + 2\, y1^4\, y3\, y4 + 5\, y1^3\, y2\, y3\, y4 + 8\, y1^2\, y2^2\, y3\, y4 + 5\, y1\, y2^3\, y3\, y4 + 2\, y2^4\, y3\, y4 + 3\, y1^3\, y3^2\, y4 + 8\, y1^2\, y2\, y3^2\, y4 + 8\, y1\, y2^2\, y3^2\, y4 + 3\, y2^3\, y3^2\, y4 + 3\, y1^2\, y3^3\, y4 + 5\, y1\, y2\, y3^3\, y4 + 3\, y2^2\, y3^3\, y4 + 2\, y1\, y3^4\, y4 + 2\, y2\, y3^4\, y4 + y3^5\, y4 + 2\, y1^4\, y4^2 + 3\, y1^3\, y2\, y4^2 + 6\, y1^2\, y2^2\, y4^2 + 3\, y1\, y2^3\, y4^2 + 2\, y2^4\, y4^2 + 3\, y1^3\, y3\, y4^2 + 8\, y1^2\, y2\, y3\, y4^2 + 8\, y1\, y2^2\, y3\, y4^2 + 3\, y2^3\, y3\, y4^2 + 6\, y1^2\, y3^2\, y4^2 + 8\, y1\, y2\, y3^2\, y4^2 + 6\, y2^2\, y3^2\, y4^2 + 3\, y1\, y3^3\, y4^2 + 3\, y2\, y3^3\, y4^2 + 2\, y3^4\, y4^2 + 2\, y1^3\, y4^3 + 3\, y1^2\, y2\, y4^3 + 3\, y1\, y2^2\, y4^3 + 2\, y2^3\, y4^3 + 3\, y1^2\, y3\, y4^3 + 5\, y1\, y2\, y3\, y4^3 + 3\, y2^2\, y3\, y4^3 + 3\, y1\, y3^2\, y4^3 + 3\, y2\, y3^2\, y4^3 + 2\, y3^3\, y4^3 + 2\, y1^2\, y4^4 + 2\, y1\, y2\, y4^4 + 2\, y2^2\, y4^4 +$$

$2 y1 y3 y4^4 + 2 y2 y3 y4^4 + 2 y3^2 y4^4 + y1 y4^5 + y2 y4^5 + y3 y4^5 + y4^6 + y1^5 y5 + 2 y1^4 y2 y5 +$
$3 y1^3 y2^2 y5 + 3 y1^2 y2^3 y5 + 2 y1 y2^4 y5 + y2^5 y5 + 2 y1^4 y3 y5 + 5 y1^3 y2 y3 y5 + 8 y1^2 y2^2 y3 y5 +$
$5 y1 y2^3 y3 y5 + 2 y2^4 y3 y5 + 3 y1^3 y3^2 y5 + 8 y1^2 y2 y3^2 y5 + 8 y1 y2^2 y3^2 y5 + 3 y2^3 y3^2 y5 +$
$3 y1^2 y3^3 y5 + 5 y1 y2 y3^3 y5 + 3 y2^2 y3^3 y5 + 2 y1 y3^4 y5 + 2 y2 y3^4 y5 + y3^5 y5 + 2 y1^4 y4 y5 +$
$5 y1^3 y2 y4 y5 + 8 y1^2 y2^2 y4 y5 + 5 y1 y2^3 y4 y5 + 2 y2^4 y4 y5 + 5 y1^3 y3 y4 y5 + 15 y1^2 y2 y3 y4 y5 +$
$15 y1 y2^2 y3 y4 y5 + 5 y2^3 y3 y4 y5 + 8 y1^2 y3^2 y4 y5 + 15 y1 y2 y3^2 y4 y5 + 8 y2^2 y3^2 y4 y5 +$
$5 y1 y3^3 y4 y5 + 5 y2 y3^3 y4 y5 + 2 y3^4 y4 y5 + 3 y1^3 y4^2 y5 + 8 y1^2 y2 y4^2 y5 + 8 y1 y2^2 y4^2 y5 +$
$3 y2^3 y4^2 y5 + 8 y1^2 y3 y4^2 y5 + 15 y1 y2 y3 y4^2 y5 + 8 y2^2 y3 y4^2 y5 + 8 y1 y3^2 y4^2 y5 +$
$8 y2 y3^2 y4^2 y5 + 3 y3^3 y4^2 y5 + 3 y1^2 y4^3 y5 + 5 y1 y2 y4^3 y5 + 3 y2^2 y4^3 y5 + 5 y1 y3 y4^3 y5 +$
$5 y2 y3 y4^3 y5 + 3 y3^2 y4^3 y5 + 2 y1 y4^4 y5 + 2 y2 y4^4 y5 + 2 y3 y4^4 y5 + y4^5 y5 + 2 y1^4 y5^2 +$
$3 y1^3 y2 y5^2 + 6 y1^2 y2^2 y5^2 + 3 y1 y2^3 y5^2 + 2 y2^4 y5^2 + 3 y1^3 y3 y5^2 + 8 y1^2 y2 y3 y5^2 +$
$8 y1 y2^2 y3 y5^2 + 3 y2^3 y3 y5^2 + 6 y1^2 y3^2 y5^2 + 8 y1 y2 y3^2 y5^2 + 6 y2^2 y3^2 y5^2 + 3 y1 y3^3 y5^2 +$
$3 y2 y3^3 y5^2 + 2 y3^4 y5^2 + 3 y1^3 y4 y5^2 + 8 y1^2 y2 y4 y5^2 + 8 y1 y2^2 y4 y5^2 + 3 y2^3 y4 y5^2 +$
$8 y1^2 y3 y4 y5^2 + 15 y1 y2 y3 y4 y5^2 + 8 y2^2 y3 y4 y5^2 + 8 y1 y3^2 y4 y5^2 + 8 y2 y3^2 y4 y5^2 +$
$3 y3^3 y4 y5^2 + 6 y1^2 y4^2 y5^2 + 8 y1 y2 y4^2 y5^2 + 6 y2^2 y4^2 y5^2 + 8 y1 y3 y4^2 y5^2 + 8 y2 y3 y4^2 y5^2 +$
$6 y3^2 y4^2 y5^2 + 3 y1 y4^3 y5^2 + 3 y2 y4^3 y5^2 + 3 y3 y4^3 y5^2 + 2 y4^4 y5^2 + 2 y1^3 y5^3 + 3 y1^2 y2 y5^3 +$
$3 y1 y2^2 y5^3 + 2 y2^3 y5^3 + 3 y1^2 y3 y5^3 + 5 y1 y2 y3 y5^3 + 3 y2^2 y3 y5^3 + 3 y1 y3^2 y5^3 +$
$3 y2 y3^2 y5^3 + 2 y3^3 y5^3 + 3 y1^2 y4 y5^3 + 5 y1 y2 y4 y5^3 + 3 y2^2 y4 y5^3 + 5 y1 y3 y4 y5^3 +$
$5 y2 y3 y4 y5^3 + 3 y3^2 y4 y5^3 + 3 y1 y4^2 y5^3 + 3 y2 y4^2 y5^3 + 3 y3 y4^2 y5^3 + 2 y4^3 y5^3 +$
$2 y1^2 y5^4 + 2 y1 y2 y5^4 + 2 y2^2 y5^4 + 2 y1 y3 y5^4 + 2 y2 y3 y5^4 + 2 y3^2 y5^4 + 2 y1 y4 y5^4 +$
$2 y2 y4 y5^4 + 2 y3 y4 y5^4 + 2 y4^2 y5^4 + y1 y5^5 + y2 y5^5 + y3 y5^5 + y4 y5^5 + y5^6 + y1^5 y6 +$
$2 y1^4 y2 y6 + 3 y1^3 y2^2 y6 + 3 y1^2 y2^3 y6 + 2 y1 y2^4 y6 + y2^5 y6 + 2 y1^4 y3 y6 + 5 y1^3 y2 y3 y6 +$
$8 y1^2 y2^2 y3 y6 + 5 y1 y2^3 y3 y6 + 2 y2^4 y3 y6 + 3 y1^3 y3^2 y6 + 8 y1^2 y2 y3^2 y6 + 8 y1 y2^2 y3^2 y6 +$
$3 y2^3 y3^2 y6 + 3 y1^2 y3^3 y6 + 5 y1 y2 y3^3 y6 + 3 y2^2 y3^3 y6 + 2 y1 y3^4 y6 + 2 y2 y3^4 y6 + y3^5 y6 +$
$2 y1^4 y4 y6 + 5 y1^3 y2 y4 y6 + 8 y1^2 y2^2 y4 y6 + 5 y1 y2^3 y4 y6 + 2 y2^4 y4 y6 + 5 y1^3 y3 y4 y6 +$
$15 y1^2 y2 y3 y4 y6 + 15 y1 y2^2 y3 y4 y6 + 5 y2^3 y3 y4 y6 + 8 y1^2 y3^2 y4 y6 + 15 y1 y2 y3^2 y4 y6 +$
$8 y2^2 y3^2 y4 y6 + 5 y1 y3^3 y4 y6 + 5 y2 y3^3 y4 y6 + 2 y3^4 y4 y6 + 3 y1^3 y4^2 y6 + 8 y1^2 y2 y4^2 y6 +$
$8 y1 y2^2 y4^2 y6 + 3 y2^3 y4^2 y6 + 8 y1^2 y3 y4^2 y6 + 15 y1 y2 y3 y4^2 y6 + 8 y2^2 y3 y4^2 y6 +$
$8 y1 y3^2 y4^2 y6 + 8 y2 y3^2 y4^2 y6 + 3 y3^3 y4^2 y6 + 3 y1^2 y4^3 y6 + 5 y1 y2 y4^3 y6 + 3 y2^2 y4^3 y6 +$
$5 y1 y3 y4^3 y6 + 5 y2 y3 y4^3 y6 + 3 y3^2 y4^3 y6 + 2 y1 y4^4 y6 + 2 y2 y4^4 y6 + 2 y3 y4^4 y6 +$
$y4^5 y6 + 2 y1^4 y5 y6 + 5 y1^3 y2 y5 y6 + 8 y1^2 y2^2 y5 y6 + 5 y1 y2^3 y5 y6 + 2 y2^4 y5 y6 +$
$5 y1^3 y3 y5 y6 + 15 y1^2 y2 y3 y5 y6 + 15 y1 y2^2 y3 y5 y6 + 5 y2^3 y3 y5 y6 + 8 y1^2 y3^2 y5 y6 +$
$15 y1 y2 y3^2 y5 y6 + 8 y2^2 y3^2 y5 y6 + 5 y1 y3^3 y5 y6 + 5 y2 y3^3 y5 y6 + 2 y3^4 y5 y6 +$
$5 y1^3 y4 y5 y6 + 15 y1^2 y2 y4 y5 y6 + 15 y1 y2^2 y4 y5 y6 + 5 y2^3 y4 y5 y6 + 15 y1^2 y3 y4 y5 y6 +$
$30 y1 y2 y3 y4 y5 y6 + 15 y2^2 y3 y4 y5 y6 + 15 y1 y3^2 y4 y5 y6 + 15 y2 y3^2 y4 y5 y6 +$
$5 y3^3 y4 y5 y6 + 8 y1^2 y4^2 y5 y6 + 15 y1 y2 y4^2 y5 y6 + 8 y2^2 y4^2 y5 y6 + 15 y1 y3 y4^2 y5 y6 +$
$15 y2 y3 y4^2 y5 y6 + 8 y3^2 y4^2 y5 y6 + 5 y1 y4^3 y5 y6 + 5 y2 y4^3 y5 y6 + 5 y3 y4^3 y5 y6 +$
$2 y4^4 y5 y6 + 3 y1^3 y5^2 y6 + 8 y1^2 y2 y5^2 y6 + 8 y1 y2^2 y5^2 y6 + 3 y2^3 y5^2 y6 + 8 y1^2 y3 y5^2 y6 +$
$15 y1 y2 y3 y5^2 y6 + 8 y2^2 y3 y5^2 y6 + 8 y1 y3^2 y5^2 y6 + 8 y2 y3^2 y5^2 y6 + 3 y3^3 y5^2 y6 +$
$8 y1^2 y4 y5^2 y6 + 15 y1 y2 y4 y5^2 y6 + 8 y2^2 y4 y5^2 y6 + 15 y1 y3 y4 y5^2 y6 + 15 y2 y3 y4 y5^2 y6 +$
$8 y3^2 y4 y5^2 y6 + 8 y1 y4^2 y5^2 y6 + 8 y2 y4^2 y5^2 y6 + 8 y3 y4^2 y5^2 y6 + 3 y4^3 y5^2 y6 +$
$3 y1^2 y5^3 y6 + 5 y1 y2 y5^3 y6 + 3 y2^2 y5^3 y6 + 5 y1 y3 y5^3 y6 + 5 y2 y3 y5^3 y6 + 3 y3^2 y5^3 y6 +$
$5 y1 y4 y5^3 y6 + 5 y2 y4 y5^3 y6 + 5 y3 y4 y5^3 y6 + 3 y4^2 y5^3 y6 + 2 y1 y5^4 y6 + 2 y2 y5^4 y6 +$

$$2\,y3\,y5^{4}\,y6 + 2\,y4\,y5^{4}\,y6 + y5^{5}\,y6 + 2\,y1^{4}\,y6^{2} + 3\,y1^{3}\,y2\,y6^{2} + 6\,y1^{2}\,y2^{2}\,y6^{2} + 3\,y1\,y2^{3}\,y6^{2} +$$
$$2\,y2^{4}\,y6^{2} + 3\,y1^{3}\,y3\,y6^{2} + 8\,y1^{2}\,y2\,y3\,y6^{2} + 8\,y1\,y2^{2}\,y3\,y6^{2} + 3\,y2^{3}\,y3\,y6^{2} + 6\,y1^{2}\,y3^{2}\,y6^{2} +$$
$$8\,y1\,y2\,y3^{2}\,y6^{2} + 6\,y2^{2}\,y3^{2}\,y6^{2} + 3\,y1\,y3^{3}\,y6^{2} + 3\,y2\,y3^{3}\,y6^{2} + 2\,y3^{4}\,y6^{2} + 3\,y1^{3}\,y4\,y6^{2} +$$
$$8\,y1^{2}\,y2\,y4\,y6^{2} + 8\,y1\,y2^{2}\,y4\,y6^{2} + 3\,y2^{3}\,y4\,y6^{2} + 8\,y1^{2}\,y3\,y4\,y6^{2} + 15\,y1\,y2\,y3\,y4\,y6^{2} +$$
$$8\,y2^{2}\,y3\,y4\,y6^{2} + 8\,y1\,y3^{2}\,y4\,y6^{2} + 8\,y2\,y3^{2}\,y4\,y6^{2} + 3\,y3^{3}\,y4\,y6^{2} + 6\,y1^{2}\,y4^{2}\,y6^{2} + 8\,y1\,y2\,y4^{2}\,y6^{2} +$$
$$6\,y2^{2}\,y4^{2}\,y6^{2} + 8\,y1\,y3\,y4^{2}\,y6^{2} + 8\,y2\,y3\,y4^{2}\,y6^{2} + 6\,y3^{2}\,y4^{2}\,y6^{2} + 3\,y1\,y4^{3}\,y6^{2} + 3\,y2\,y4^{3}\,y6^{2} +$$
$$3\,y3\,y4^{3}\,y6^{2} + 2\,y4^{4}\,y6^{2} + 3\,y1^{3}\,y5\,y6^{2} + 8\,y1^{2}\,y2\,y5\,y6^{2} + 8\,y1\,y2^{2}\,y5\,y6^{2} + 3\,y2^{3}\,y5\,y6^{2} +$$
$$8\,y1^{2}\,y3\,y5\,y6^{2} + 15\,y1\,y2\,y3\,y5\,y6^{2} + 8\,y2^{2}\,y3\,y5\,y6^{2} + 8\,y1\,y3^{2}\,y5\,y6^{2} + 8\,y2\,y3^{2}\,y5\,y6^{2} +$$
$$3\,y3^{3}\,y5\,y6^{2} + 8\,y1^{2}\,y4\,y5\,y6^{2} + 15\,y1\,y2\,y4\,y5\,y6^{2} + 8\,y2^{2}\,y4\,y5\,y6^{2} + 15\,y1\,y3\,y4\,y5\,y6^{2} +$$
$$15\,y2\,y3\,y4\,y5\,y6^{2} + 8\,y3^{2}\,y4\,y5\,y6^{2} + 8\,y1\,y4^{2}\,y5\,y6^{2} + 8\,y2\,y4^{2}\,y5\,y6^{2} + 8\,y3\,y4^{2}\,y5\,y6^{2} +$$
$$3\,y4^{3}\,y5\,y6^{2} + 6\,y1^{2}\,y5^{2}\,y6^{2} + 8\,y1\,y2\,y5^{2}\,y6^{2} + 6\,y2^{2}\,y5^{2}\,y6^{2} + 8\,y1\,y3\,y5^{2}\,y6^{2} + 8\,y2\,y3\,y5^{2}\,y6^{2} +$$
$$6\,y3^{2}\,y5^{2}\,y6^{2} + 8\,y1\,y4\,y5^{2}\,y6^{2} + 8\,y2\,y4\,y5^{2}\,y6^{2} + 8\,y3\,y4\,y5^{2}\,y6^{2} + 6\,y4^{2}\,y5^{2}\,y6^{2} + 3\,y1\,y5^{3}\,y6^{2} +$$
$$3\,y2\,y5^{3}\,y6^{2} + 3\,y3\,y5^{3}\,y6^{2} + 3\,y4\,y5^{3}\,y6^{2} + 2\,y5^{4}\,y6^{2} + 2\,y1^{3}\,y6^{3} + 3\,y1^{2}\,y2\,y6^{3} + 3\,y1\,y2^{2}\,y6^{3} +$$
$$2\,y2^{3}\,y6^{3} + 3\,y1^{2}\,y3\,y6^{3} + 5\,y1\,y2\,y3\,y6^{3} + 3\,y2^{2}\,y3\,y6^{3} + 3\,y1\,y3^{2}\,y6^{3} + 3\,y2\,y3^{2}\,y6^{3} +$$
$$2\,y3^{3}\,y6^{3} + 3\,y1^{2}\,y4\,y6^{3} + 5\,y1\,y2\,y4\,y6^{3} + 3\,y2^{2}\,y4\,y6^{3} + 5\,y1\,y3\,y4\,y6^{3} + 5\,y2\,y3\,y4\,y6^{3} +$$
$$3\,y3^{2}\,y4\,y6^{3} + 3\,y1\,y4^{2}\,y6^{3} + 3\,y2\,y4^{2}\,y6^{3} + 3\,y3\,y4^{2}\,y6^{3} + 2\,y4^{3}\,y6^{3} + 3\,y1^{2}\,y5\,y6^{3} +$$
$$5\,y1\,y2\,y5\,y6^{3} + 3\,y2^{2}\,y5\,y6^{3} + 5\,y1\,y3\,y5\,y6^{3} + 5\,y2\,y3\,y5\,y6^{3} + 3\,y3^{2}\,y5\,y6^{3} + 5\,y1\,y4\,y5\,y6^{3} +$$
$$5\,y2\,y4\,y5\,y6^{3} + 5\,y3\,y4\,y5\,y6^{3} + 3\,y4^{2}\,y5\,y6^{3} + 3\,y1\,y5^{2}\,y6^{3} + 3\,y2\,y5^{2}\,y6^{3} + 3\,y3\,y5^{2}\,y6^{3} +$$
$$3\,y4\,y5^{2}\,y6^{3} + 2\,y5^{3}\,y6^{3} + 2\,y1^{2}\,y6^{4} + 2\,y1\,y2\,y6^{4} + 2\,y2^{2}\,y6^{4} + 2\,y1\,y3\,y6^{4} + 2\,y2\,y3\,y6^{4} +$$
$$2\,y3^{2}\,y6^{4} + 2\,y1\,y4\,y6^{4} + 2\,y2\,y4\,y6^{4} + 2\,y3\,y4\,y6^{4} + 2\,y4^{2}\,y6^{4} + 2\,y1\,y5\,y6^{4} + 2\,y2\,y5\,y6^{4} +$$
$$2\,y3\,y5\,y6^{4} + 2\,y4\,y5\,y6^{4} + 2\,y5^{2}\,y6^{4} + y1\,y6^{5} + y2\,y6^{5} + y3\,y6^{5} + y4\,y6^{5} + y5\,y6^{5} + y6^{6}$$

In[13]:= **Length[Expand[FG2]]**

Out[13]= 462

In[14]:= **Coefficient[FG2, y1 y2 y3 y4 y5 y6]**

Out[14]= 30

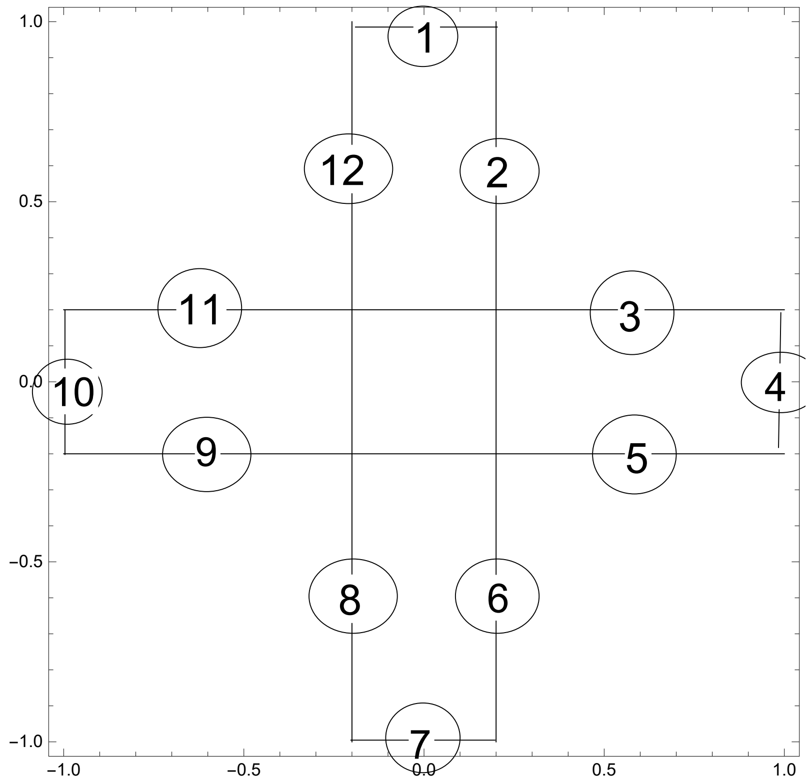
In[15]:= **Coefficient[FG2, $y4^2\,y6^4$]**

Out[15]= 2

Problem 3

Suppose a medical relief agency plans to design a symbol for their organization in the shape of a regular cross, as in the figure below. To symbolize the purpose of the organization and emphasize its international constituency, its board of directors decides that the cross should be white in color, with each of the twelve line segments outlining the cross colored red, green, blue, or yellow, with an equal number of lines of each colour. If we discount rotations and flips, how many different ways are there to design the symbol?

We use Mathematica for drawing the cross



Answer

- See pdf file.

Obtain the cycle index polynomial:

```
In[16]:= PG3[x1_, x2_, x3_, x4_, x5_, x6_, x7_, x8_, x9_, x10_, x11_, x12_] :=
  (x1^12 + 2 x4^3 + 2 x1^2 x2^5 + 3 x2^6) / 8
```

For a colouring of 12 edges with four colours r, g, b, y we have the pattern inventory polynomial:

```
In[17]:= FG3 = Expand[PG3[r + g + b + y, r^2 + g^2 + b^2 + y^2, r^3 + g^3 + b^3 + y^3,
    r^4 + g^4 + b^4 + y^4, r^5 + g^5 + b^5 + y^5, r^6 + g^6 + b^6 + y^6,
    r^7 + g^7 + b^7 + y^7, r^8 + g^8 + b^8 + y^8, r^9 + g^9 + b^9 + y^9,
    r^10 + g^10 + b^10 + y^10, r^11 + g^11 + b^11 + y^11, r^12 + g^12 + b^12 + y^12]]
```

Out[17]=

$$
\begin{aligned}
& b^{12} + 2 b^{11} g + 12 b^{10} g^2 + 30 b^9 g^3 + 72 b^8 g^4 + 104 b^7 g^5 + 128 b^6 g^6 + 104 b^5 g^7 + 72 b^4 g^8 + 30 b^3 g^9 + \\
& 12 b^2 g^{10} + 2 b g^{11} + g^{12} + 2 b^{11} r + 17 b^{10} g r + 85 b^9 g^2 r + 250 b^8 g^3 r + 500 b^7 g^4 r + 698 b^6 g^5 r + \\
& 698 b^5 g^6 r + 500 b^4 g^7 r + 250 b^3 g^8 r + 85 b^2 g^9 r + 17 b g^{10} r + 2 g^{11} r + 12 b^{10} r^2 + 85 b^9 g r^2 + \\
& 390 b^8 g^2 r^2 + 1000 b^7 g^3 r^2 + 1770 b^6 g^4 r^2 + 2094 b^5 g^5 r^2 + 1770 b^4 g^6 r^2 + 1000 b^3 g^7 r^2 + \\
& 390 b^2 g^8 r^2 + 85 b g^9 r^2 + 12 g^{10} r^2 + 30 b^9 r^3 + 250 b^8 g r^3 + 1000 b^7 g^2 r^3 + 2320 b^6 g^3 r^3 + \\
& 3480 b^5 g^4 r^3 + 3480 b^4 g^5 r^3 + 2320 b^3 g^6 r^3 + 1000 b^2 g^7 r^3 + 250 b g^8 r^3 + 30 g^9 r^3 + 72 b^8 r^4 + \\
& 500 b^7 g r^4 + 1770 b^6 g^2 r^4 + 3480 b^5 g^3 r^4 + 4389 b^4 g^4 r^4 + 3480 b^3 g^5 r^4 + 1770 b^2 g^6 r^4 + \\
& 500 b g^7 r^4 + 72 g^8 r^4 + 104 b^7 r^5 + 698 b^6 g r^5 + 2094 b^5 g^2 r^5 + 3480 b^4 g^3 r^5 + 3480 b^3 g^4 r^5 + \\
& 2094 b^2 g^5 r^5 + 698 b g^6 r^5 + 104 g^7 r^5 + 128 b^6 r^6 + 698 b^5 g r^6 + 1770 b^4 g^2 r^6 + 2320 b^3 g^3 r^6 + \\
& 1770 b^2 g^4 r^6 + 698 b g^5 r^6 + 128 g^6 r^6 + 104 b^5 r^7 + 500 b^4 g r^7 + 1000 b^3 g^2 r^7 + 1000 b^2 g^3 r^7 +
\end{aligned}
$$

$500 b g^4 r^7 + 104 g^5 r^7 + 72 b^4 r^8 + 250 b^3 g r^8 + 390 b^2 g^2 r^8 + 250 b g^3 r^8 + 72 g^4 r^8 + 30 b^3 r^9 +$
$85 b^2 g r^9 + 85 b g^2 r^9 + 30 g^3 r^9 + 12 b^2 r^{10} + 17 b g r^{10} + 12 g^2 r^{10} + 2 b r^{11} + 2 g r^{11} +$
$r^{12} + 2 b^{11} y + 17 b^{10} g y + 85 b^9 g^2 y + 250 b^8 g^3 y + 500 b^7 g^4 y + 698 b^6 g^5 y + 698 b^5 g^6 y +$
$500 b^4 g^7 y + 250 b^3 g^8 y + 85 b^2 g^9 y + 17 b g^{10} y + 2 g^{11} y + 17 b^{10} r y + 165 b^9 g r y +$
$745 b^8 g^2 r y + 1980 b^7 g^3 r y + 3470 b^6 g^4 r y + 4158 b^5 g^5 r y + 3470 b^4 g^6 r y + 1980 b^3 g^7 r y +$
$745 b^2 g^8 r y + 165 b g^9 r y + 17 g^{10} r y + 85 b^9 r^2 y + 745 b^8 g r^2 y + 2980 b^7 g^2 r^2 y +$
$6940 b^6 g^3 r^2 y + 10410 b^5 g^4 r^2 y + 10410 b^4 g^5 r^2 y + 6940 b^3 g^6 r^2 y + 2980 b^2 g^7 r^2 y +$
$745 b g^8 r^2 y + 85 g^9 r^2 y + 250 b^8 r^3 y + 1980 b^7 g r^3 y + 6940 b^6 g^2 r^3 y + 13860 b^5 g^3 r^3 y +$
$17340 b^4 g^4 r^3 y + 13860 b^3 g^5 r^3 y + 6940 b^2 g^6 r^3 y + 1980 b g^7 r^3 y + 250 g^8 r^3 y + 500 b^7 r^4 y +$
$3470 b^6 g r^4 y + 10410 b^5 g^2 r^4 y + 17340 b^4 g^3 r^4 y + 17340 b^3 g^4 r^4 y + 10410 b^2 g^5 r^4 y +$
$3470 b g^6 r^4 y + 500 g^7 r^4 y + 698 b^6 r^5 y + 4158 b^5 g r^5 y + 10410 b^4 g^2 r^5 y + 13860 b^3 g^3 r^5 y +$
$10410 b^2 g^4 r^5 y + 4158 b g^5 r^5 y + 698 g^6 r^5 y + 698 b^5 r^6 y + 3470 b^4 g r^6 y + 6940 b^3 g^2 r^6 y +$
$6940 b^2 g^3 r^6 y + 3470 b g^4 r^6 y + 698 g^5 r^6 y + 500 b^4 r^7 y + 1980 b^3 g r^7 y + 2980 b^2 g^2 r^7 y +$
$1980 b g^3 r^7 y + 500 g^4 r^7 y + 250 b^3 r^8 y + 745 b^2 g r^8 y + 745 b g^2 r^8 y + 250 g^3 r^8 y +$
$85 b^2 r^9 y + 165 b g r^9 y + 85 g^2 r^9 y + 17 b r^{10} y + 17 g r^{10} y + 2 r^{11} y + 12 b^{10} y^2 + 85 b^9 g y^2 +$
$390 b^8 g^2 y^2 + 1000 b^7 g^3 y^2 + 1770 b^6 g^4 y^2 + 2094 b^5 g^5 y^2 + 1770 b^4 g^6 y^2 + 1000 b^3 g^7 y^2 +$
$390 b^2 g^8 y^2 + 85 b g^9 y^2 + 12 g^{10} y^2 + 85 b^9 r y^2 + 745 b^8 g r y^2 + 2980 b^7 g^2 r y^2 + 6940 b^6 g^3 r y^2 +$
$10410 b^5 g^4 r y^2 + 10410 b^4 g^5 r y^2 + 6940 b^3 g^6 r y^2 + 2980 b^2 g^7 r y^2 + 745 b g^8 r y^2 + 85 g^9 r y^2 +$
$390 b^8 r^2 y^2 + 2980 b^7 g r^2 y^2 + 10470 b^6 g^2 r^2 y^2 + 20820 b^5 g^3 r^2 y^2 + 26100 b^4 g^4 r^2 y^2 +$
$20820 b^3 g^5 r^2 y^2 + 10470 b^2 g^6 r^2 y^2 + 2980 b g^7 r^2 y^2 + 390 g^8 r^2 y^2 + 1000 b^7 r^3 y^2 +$
$6940 b^6 g r^3 y^2 + 20820 b^5 g^2 r^3 y^2 + 34680 b^4 g^3 r^3 y^2 + 34680 b^3 g^4 r^3 y^2 + 20820 b^2 g^5 r^3 y^2 +$
$6940 b g^6 r^3 y^2 + 1000 g^7 r^3 y^2 + 1770 b^6 r^4 y^2 + 10410 b^5 g r^4 y^2 + 26100 b^4 g^2 r^4 y^2 +$
$34680 b^3 g^3 r^4 y^2 + 26100 b^2 g^4 r^4 y^2 + 10410 b g^5 r^4 y^2 + 1770 g^6 r^4 y^2 + 2094 b^5 r^5 y^2 +$
$10410 b^4 g r^5 y^2 + 20820 b^3 g^2 r^5 y^2 + 20820 b^2 g^3 r^5 y^2 + 10410 b g^4 r^5 y^2 + 2094 g^5 r^5 y^2 +$
$1770 b^4 r^6 y^2 + 6940 b^3 g r^6 y^2 + 10470 b^2 g^2 r^6 y^2 + 6940 b g^3 r^6 y^2 + 1770 g^4 r^6 y^2 +$
$1000 b^3 r^7 y^2 + 2980 b^2 g r^7 y^2 + 2980 b g^2 r^7 y^2 + 1000 g^3 r^7 y^2 + 390 b^2 r^8 y^2 + 745 b g r^8 y^2 +$
$390 g^2 r^8 y^2 + 85 b r^9 y^2 + 85 g r^9 y^2 + 12 r^{10} y^2 + 30 b^9 y^3 + 250 b^8 g y^3 + 1000 b^7 g^2 y^3 +$
$2320 b^6 g^3 y^3 + 3480 b^5 g^4 y^3 + 3480 b^4 g^5 y^3 + 2320 b^3 g^6 y^3 + 1000 b^2 g^7 y^3 + 250 b g^8 y^3 +$
$30 g^9 y^3 + 250 b^8 r y^3 + 1980 b^7 g r y^3 + 6940 b^6 g^2 r y^3 + 13860 b^5 g^3 r y^3 + 17340 b^4 g^4 r y^3 +$
$13860 b^3 g^5 r y^3 + 6940 b^2 g^6 r y^3 + 1980 b g^7 r y^3 + 250 g^8 r y^3 + 1000 b^7 r^2 y^3 + 6940 b^6 g r^2 y^3 +$
$20820 b^5 g^2 r^2 y^3 + 34680 b^4 g^3 r^2 y^3 + 34680 b^3 g^4 r^2 y^3 + 20820 b^2 g^5 r^2 y^3 + 6940 b g^6 r^2 y^3 +$
$1000 g^7 r^2 y^3 + 2320 b^6 r^3 y^3 + 13860 b^5 g r^3 y^3 + 34680 b^4 g^2 r^3 y^3 + 46200 b^3 g^3 r^3 y^3 +$
$34680 b^2 g^4 r^3 y^3 + 13860 b g^5 r^3 y^3 + 2320 g^6 r^3 y^3 + 3480 b^5 r^4 y^3 + 17340 b^4 g r^4 y^3 +$
$34680 b^3 g^2 r^4 y^3 + 34680 b^2 g^3 r^4 y^3 + 17340 b g^4 r^4 y^3 + 3480 g^5 r^4 y^3 + 3480 b^4 r^5 y^3 +$
$13860 b^3 g r^5 y^3 + 20820 b^2 g^2 r^5 y^3 + 13860 b g^3 r^5 y^3 + 3480 g^4 r^5 y^3 + 2320 b^3 r^6 y^3 +$
$6940 b^2 g r^6 y^3 + 6940 b g^2 r^6 y^3 + 2320 g^3 r^6 y^3 + 1000 b^2 r^7 y^3 + 1980 b g r^7 y^3 + 1000 g^2 r^7 y^3 +$
$250 b r^8 y^3 + 250 g r^8 y^3 + 30 r^9 y^3 + 72 b^8 y^4 + 500 b^7 g y^4 + 1770 b^6 g^2 y^4 + 3480 b^5 g^3 y^4 +$
$4389 b^4 g^4 y^4 + 3480 b^3 g^5 y^4 + 1770 b^2 g^6 y^4 + 500 b g^7 y^4 + 72 g^8 y^4 + 500 b^7 r y^4 + 3470 b^6 g r y^4 +$
$10410 b^5 g^2 r y^4 + 17340 b^4 g^3 r y^4 + 17340 b^3 g^4 r y^4 + 10410 b^2 g^5 r y^4 + 3470 b g^6 r y^4 +$
$500 g^7 r y^4 + 1770 b^6 r^2 y^4 + 10410 b^5 g r^2 y^4 + 26100 b^4 g^2 r^2 y^4 + 34680 b^3 g^3 r^2 y^4 +$
$26100 b^2 g^4 r^2 y^4 + 10410 b g^5 r^2 y^4 + 1770 g^6 r^2 y^4 + 3480 b^5 r^3 y^4 + 17340 b^4 g r^3 y^4 +$
$34680 b^3 g^2 r^3 y^4 + 34680 b^2 g^3 r^3 y^4 + 17340 b g^4 r^3 y^4 + 3480 g^5 r^3 y^4 + 4389 b^4 r^4 y^4 +$
$17340 b^3 g r^4 y^4 + 26100 b^2 g^2 r^4 y^4 + 17340 b g^3 r^4 y^4 + 4389 g^4 r^4 y^4 + 3480 b^3 r^5 y^4 +$

$$10\,410\, b^2 g r^5 y^4 + 10\,410\, b g^2 r^5 y^4 + 3480\, g^3 r^5 y^4 + 1770\, b^2 r^6 y^4 + 3470\, b g r^6 y^4 + 1770\, g^2 r^6 y^4 +$$
$$500\, b r^7 y^4 + 500\, g r^7 y^4 + 72\, r^8 y^4 + 104\, b^7 y^5 + 698\, b^6 g y^5 + 2094\, b^5 g^2 y^5 + 3480\, b^4 g^3 y^5 +$$
$$3480\, b^3 g^4 y^5 + 2094\, b^2 g^5 y^5 + 698\, b g^6 y^5 + 104\, g^7 y^5 + 698\, b^6 r y^5 + 4158\, b^5 g r y^5 +$$
$$10\,410\, b^4 g^2 r y^5 + 13\,860\, b^3 g^3 r y^5 + 10\,410\, b^2 g^4 r y^5 + 4158\, b g^5 r y^5 + 698\, g^6 r y^5 +$$
$$2094\, b^5 r^2 y^5 + 10\,410\, b^4 g r^2 y^5 + 20\,820\, b^3 g^2 r^2 y^5 + 20\,820\, b^2 g^3 r^2 y^5 + 10\,410\, b g^4 r^2 y^5 +$$
$$2094\, g^5 r^2 y^5 + 3480\, b^4 r^3 y^5 + 13\,860\, b^3 g r^3 y^5 + 20\,820\, b^2 g^2 r^3 y^5 + 13\,860\, b g^3 r^3 y^5 +$$
$$3480\, g^4 r^3 y^5 + 3480\, b^3 r^4 y^5 + 10\,410\, b^2 g r^4 y^5 + 10\,410\, b g^2 r^4 y^5 + 3480\, g^3 r^4 y^5 + 2094\, b^2 r^5 y^5 +$$
$$4158\, b g r^5 y^5 + 2094\, g^2 r^5 y^5 + 698\, b r^6 y^5 + 698\, g r^6 y^5 + 104\, r^7 y^5 + 128\, b^6 y^6 + 698\, b^5 g y^6 +$$
$$1770\, b^4 g^2 y^6 + 2320\, b^3 g^3 y^6 + 1770\, b^2 g^4 y^6 + 698\, b g^5 y^6 + 128\, g^6 y^6 + 698\, b^5 r y^6 + 3470\, b^4 g r y^6 +$$
$$6940\, b^3 g^2 r y^6 + 6940\, b^2 g^3 r y^6 + 3470\, b g^4 r y^6 + 698\, g^5 r y^6 + 1770\, b^4 r^2 y^6 + 6940\, b^3 g r^2 y^6 +$$
$$10\,470\, b^2 g^2 r^2 y^6 + 6940\, b g^3 r^2 y^6 + 1770\, g^4 r^2 y^6 + 2320\, b^3 r^3 y^6 + 6940\, b^2 g r^3 y^6 +$$
$$6940\, b g^2 r^3 y^6 + 2320\, g^3 r^3 y^6 + 1770\, b^2 r^4 y^6 + 3470\, b g r^4 y^6 + 1770\, g^2 r^4 y^6 + 698\, b r^5 y^6 +$$
$$698\, g r^5 y^6 + 128\, r^6 y^6 + 104\, b^5 y^7 + 500\, b^4 g y^7 + 1000\, b^3 g^2 y^7 + 1000\, b^2 g^3 y^7 + 500\, b g^4 y^7 +$$
$$104\, g^5 y^7 + 500\, b^4 r y^7 + 1980\, b^3 g r y^7 + 2980\, b^2 g^2 r y^7 + 1980\, b g^3 r y^7 + 500\, g^4 r y^7 +$$
$$1000\, b^3 r^2 y^7 + 2980\, b^2 g r^2 y^7 + 2980\, b g^2 r^2 y^7 + 1000\, g^3 r^2 y^7 + 1000\, b^2 r^3 y^7 + 1980\, b g r^3 y^7 +$$
$$1000\, g^2 r^3 y^7 + 500\, b r^4 y^7 + 500\, g r^4 y^7 + 104\, r^5 y^7 + 72\, b^4 y^8 + 250\, b^3 g y^8 + 390\, b^2 g^2 y^8 +$$
$$250\, b g^3 y^8 + 72\, g^4 y^8 + 250\, b^3 r y^8 + 745\, b^2 g r y^8 + 745\, b g^2 r y^8 + 250\, g^3 r y^8 + 390\, b^2 r^2 y^8 +$$
$$745\, b g r^2 y^8 + 390\, g^2 r^2 y^8 + 250\, b r^3 y^8 + 250\, g r^3 y^8 + 72\, r^4 y^8 + 30\, b^3 y^9 + 85\, b^2 g y^9 +$$
$$85\, b g^2 y^9 + 30\, g^3 y^9 + 85\, b^2 r y^9 + 165\, b g r y^9 + 85\, g^2 r y^9 + 85\, b r^2 y^9 + 85\, g r^2 y^9 + 30\, r^3 y^9 +$$
$$12\, b^2 y^{10} + 17\, b g y^{10} + 12\, g^2 y^{10} + 17\, b r y^{10} + 17\, g r y^{10} + 12\, r^2 y^{10} + 2\, b y^{11} + 2\, g y^{11} + 2\, r y^{11} + y^{12}$$

In[18]:= **Length[FG3]**

Out[18]= 455

In[19]:= **Coefficient[FG3, r^3 g^3 b^3 y^3]**

Out[19]= 46 200