# Graph Theory

## Introduction. Distance in Graphs. Trees

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# What is Graph Theory?

- The study of graphs as mathematical structures $G = (V, E)$ used to model pairwise relations (*a.k.a.* edges) between objects of a collection $V$.
  - The objects are modeled as nodes (or vertices) of a set $V$
  - The pairwise relations are modeled as edges, which are elements of a set $E$.
- Graphs differ mainly by the types of edges between nodes. Most common types of graphs are:
  - **Undirected:** there is no distinction between the nodes associated with each edge.
  - **Directed:** edges are arcs from one node to another.
  - **Weighted:** every edge has a weight which is typically a real number.
  - **Labeled:** every edge has its own label.

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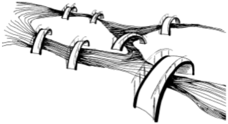
**Graphs are among the most frequently used models in problem solving.**

# History of graph theory

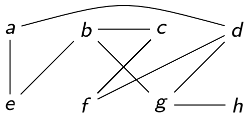
- 1736: L. Euler publishes "Seven Bridges of Königsberg" – first paper on graph theory.



  - Later: Euler's formula relating the number of edges, vertices, and faces of a convex polyhedron ⇒ generalizations by Cauchy and L'Huillier ⇒ study of topology and special classes of graphs.
- 1852: De Morgan introduces the "Four Color Map Conjecture": four is the minimum number of colors required to color any map where bordering regions are colored differently.
  - 1969: Heesch publishes a solving method
  - 1976: computer-generated proof of K. Appel and W. Haken.
- 1878: The term "graph" was first used by Sylvester in a publication in *Nature*.
- 1936: D. König publishes first textbook on graph theory.

ASSUMPTION: $G = (V, E)$ is a simple graph or digraph.

- The order of $G$ is $|V|$, the number of its nodes.
- The size of $G$ is $|E|$, the number of its edges.
- The neighborhood of $v \in V$ is $N(v) = \{x \in V \mid (v, x) \in E\}$.
- The closed neighborhood of $v \in V$ is $N[v] = \{v\} \cup N(v)$.
- The degree of $v \in V$ is the number of edges incident with $V$: $\deg(v) = |\{e \in E \mid e = (v, x) \text{ or } e = (x, v) \text{ for some } x \in V\}|$
- The maximum degree of $G$ is $\Delta(G) = \max\{\deg(v) \mid v \in V\}$.
- The minimum degree of $G$ is $\delta(G) = \min\{\deg(v) \mid v \in V\}$.
- The degree sequence of $G$ with order $n$ is the $n$-term sequence (usually written in descending order) of the vertex degrees of its nodes.




$G = (V, E)$ where $V = \{a, b, c, d, e, f, g, h\}$, $E = \{\{a, d\}, \{a, e\}, \{b, c\}, \{b, e\}, \{b, g\}, \{c, f\}, \{d, f\}, \{d, g\}, \{g, h\}\}$

- $N(d) = \{a, f, g\}$, $N[d] = \{a, d, f, g\}$,
- $\Delta(G) = \deg(b) = 3$
- $\delta(G) = \deg(h) = 1$,
- The degree sequence is $3, 3, 3, 2, 2, 2, 2, 1$

# First Theorem of Graph Theory

**Theorem**

In a graph $G$, the sum of the degrees of the vertices is equal to twice the number of edges. Consequently, the number of vertices with odd degree is even.

COMBINATORIAL PROOF.

Let $S = \sum_{v \in V} \deg(v)$. Notice that in counting $S$, we count each edge exactly twice. Thus, $S = 2|E|$ (the sum of the degrees is twice the number of edges). Since

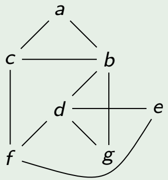
$$S = \sum_{\substack{v \in V \\ deg(v) \text{ even}}} deg(v) + \sum_{\substack{v \in V \\ deg(v) \text{ odd}}} deg(v)$$

and $S$ is even, the second sum must be even, thus the number of vertices with odd degree is even.

ASSUMPTION: $G = (V, E)$ is a simple graph or digraph.

- A walk or path in $G$ is a sequence of (not necessarily distinct) nodes $v_1, v_2, \ldots, v_k$ such that $(v_i, v_{i+1}) \in E$ for $i = 1, 2, \ldots, k-1$. Such a walk is sometimes called a $v_1 - v_k$ walk.
    - $v_1$ and $v_k$ are the end vertices of the walk.
    - If the vertices in a walk are distinct, then the walk is called a simple path.
    - If the edges in a walk are distinct, then the walk is called a trail.
- A cycle is a simple path $v_1, \ldots, v_k$ (where $k \geq 3$) together with the edge $(v_k, v_1)$.
- A circuit or closed trail is a trail that begins and ends at the same node.
- The length of a walk (or simple path, trail, cycle, circuit) is its number of edges, counting repetitions.

## Example




- $a, c, f, c, b, d$ is a walk of length 5.
- $b, a, c, b, d$ is a trail of length 4.
- $d, g, b, a, c, f, e$ is a simple path of length 6.
- $g, d, b, c, a, b, g$ is a circuit.
- $e, d, b, a, c, f, e$ is a cycle.

Note that walks, trails and simple paths can have length 0. The minimum length of a cycle or circuit is 3.

**Theorem**

In a graph $G$ with vertices $u$ and $v$, every $u - v$ walk contains a $u - v$ simple path.

PROOF. Let $W$ be a $u - v$ walk in $G$. We prove this theorem by induction on the length of the walk $W$.

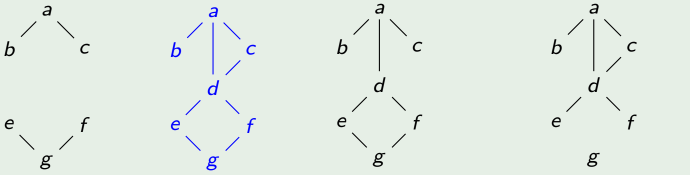
- If $W$ has length 1 or 2, then it is easy to see that $W$ must be a simple path.
- For the induction hypothesis, suppose the result is true for all walks of length $< k$ and suppose $W$ has length $k$. Say that $W$ is $u = w_0, w_1, \ldots, w_{k-1}, w_k = v$. If the nodes are distinct, then $W$ itself is the desired $u - v$ simple path. If not, then let $j$ be the smallest integer such that $w_j = w_r$ for some $r > j$. Let $W_1$ be the walk $u = w_0, \ldots, w_j, w_{r+1} \ldots, w_k = v$. This walk has length strictly less than $k$, and thus $W_1$ contains a $u - v$ simple path by induction hypothesis. Thus $W$ contains a simple $u - v$ path.

# Operations on graphs

ASSUMPTIONS: $G = (V, E)$ is a simple graph,
$v \in V$, $S \subseteq V$, $e \in E$, $T \subseteq E$

- Vertex deletion:
  - $G - v$ is the graph obtained by removing $v$ and all edges incident with $v$ from $G$.
  - $G - S$ is the graph obtained by removing each node of $S$ and each edge incident with a node of $S$ from $G$.
- Edge deletion:
  - $G - e$ is the graph obtained by removing only the edge $e$ from $G$ (its end nodes stay).
  - $G - T$ is the graph obtained by removing each edge of $T$ from $G$.
- $G$ is connected if every pair of nodes can be joined by a path. Otherwise, $G$ is disconnected.
- A component of $G$ is a maximal connected piece of $G$.
- $v$ is a cut vertex if $G - v$ has more components than $G$.
- $e$ is a bridge if $G - e$ has more components than $G$.

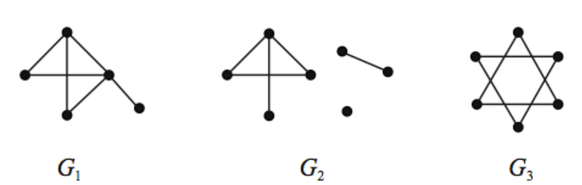
## Example (Deletion operations)




$G-d$ $\quad$ $G$ $\quad$ $G-(c,d)$ $\quad$ $G-\{(e,g),(f,g)\}$

$d$ is a cut node in $G$. $(a,b)$ is a bridge in $G$.

## Example (Connected and disconnected graphs)



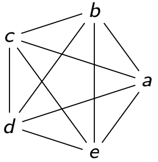
ASSUMPTION: $G = (V, E)$ is a graph.

- $\emptyset \neq S \subsetneq V$ is a node cut set of $G$ if $G - S$ is disconnected.
- $G$ is complete if every node is adjacent to every other node. We write $K_n$ for the complete graph with $n$ nodes.
  - The complete graphs $K_n$ have no node cut sets because $K_n - S$ is connected for all proper subsets $S$ of the set of nodes.
- If $G$ is not complete then the connectivity of $G$, denoted by $\kappa(G)$, is the minimum size of a node cut set of $G$.
  - If $G$ is a connected and incomplete graph of order $n$, then $1 \leq \kappa(G) \leq n - 2$.
  - If $G$ is disconnected, then $\kappa(G) = 0$.
  - If $G = K_n$ then we say that $\kappa(G) = n - 1$.
- If $k > 0$, we say that $G$ is $k$-connected if $k \leq \kappa(G)$.

# Consequences of the definitions

1. A graph is connected if and only if $\kappa(G) \geq 1$.
2. $\kappa(G) \geq 2$ if and only if $G$ is connected and has no cycles.
3. Every 2-connected graph contains at least one cycle.
4. For every graph $G$, $\kappa(G) \leq \delta(G)$.

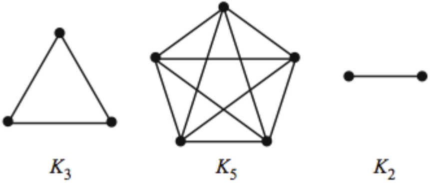
1. If $G$ is a graph of order $n$, what is the maximum number of edges in $G$?
2. Prove that for any graph $G$ of order at least 2, the degree sequence has at least one pair of repeated entries.
3. Consider the complete graph $K_5$ shown in the following figure.




   a. How many different simple paths have $c$ as an end vertex?
   b. How many different simple paths avoid vertex $c$ altogether?
   c. What is the maximum length of a circuit in this graph? Give an example of such a circuit.

4. Let $G$ be a graph where $\delta(G) \geq k$.
   a. Prove that $G$ has a simple path of length at least $k$.
   b. If $k \geq 2$, prove that $G$ has a cycle of length at least $k+1$.
5. Prove that every closed odd walk in a graph contains an odd cycle.
6. Let $P_1$ and $P_2$ be two paths of maximum length in a connected graph $G$. Prove that $P_1$ and $P_2$ have a common vertex.
7. Prove that every 2-connected graph contains at least one cycle.

1. The complete graphs $K_n$. The graph $K_n$ has order $n$ and a connection between every two nodes. Examples:

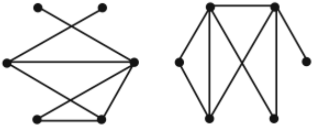



2. Empty graphs $E_n$. The graph $E_n$ has order $n$ and no edges. Example:

# Special types of graphs
## Complements and regular graphs

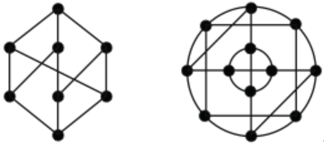
ASSUMPTION: $G = (V, E)$ is a graph.

- The complement of $G$ is the graph $\overline{G}$ whose node set is the same as that of $G$ and whose
  edge set consists of all the edges that are not in $E$. For example

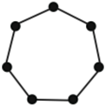


$G$ $\overline{G}$

- $G$ is regular if all its nodes have the same degree. $G$ is $r$-regular if $\deg(v) = r$ for all nodes $v$ in $G$.
  $K_n$ are $(n-1)$-regular graphs; $E_n$ are 0-regular graphs.



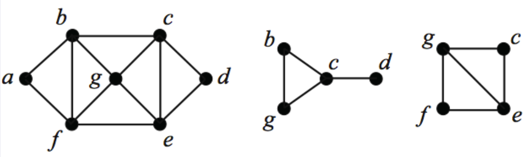
- The cycle $C_n$ is simply a cycle on $n$ vertices. Example: The graph $C_7$ looks as follows:



- The graph $P_n$ is a simple path on $n$ vertices. For example, the graph $P_6$ looks as follows:

- Given a graph $G = (V, E)$ and a subset $S \subseteq V$, the subgraph of $G$ induced by $S$, denoted $\langle S \rangle_G$, is the subgraph with vertex set $S$ and with edge set $\{(u, v) | u, v \in S \text{ and } (u, v) \in E\}$. So, $\langle S \rangle_G$ contains all vertices of $S$ and all edges of $G$ whose end vertices are both in $S$.

**Example: a graph and two of its induced subgraphs**

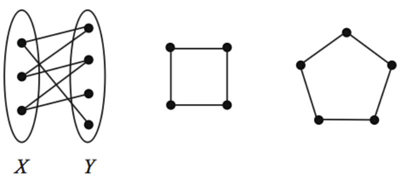



- A graph $G = (V, E)$ is bipartite if $V$ can be partitioned into two sets $X$ and $Y$ such that every edge of $G$ has one end vertex in $X$ and the other in $Y$.
  - In this case, $X$ and $Y$ are called the partite sets.
- A complete bipartite graph is a bipartite graph with partite sets $X$ and $Y$ such that its edge set is $E = \{(x, y) \mid x \in X, y \in Y\}$.
  - Such a graph is complete bipartite graph denoted by $K_{|X|,|Y|}$.

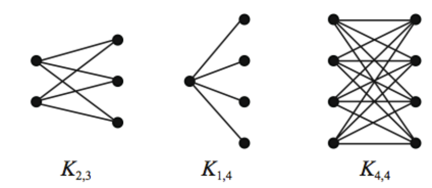
# Special types of graphs: Examples

## Bipartite graphs

The first two graphs in the following figure are bipartite, whereas the third graph is not bipartite.




## Complete bipartite graphs



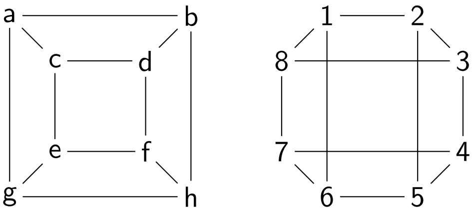
**Theorem**

A graph with at least 2 nodes is bipartite if and only if it contains no odd cycles.

PROOF.
"$\Rightarrow$:" Let $G = (V, E)$ be a bipartite graph with partite sets $X$ and $Y$, and let $C = v_1, \ldots, v_k, v_1$ be a cycle in $G$. We can assume $v_1 \in X$ w.l.o.g. Then $v_i \in X$ for all even $i$ and $v_i \in Y$ for all odd $i$. Since $(v_k, v_1) \in E$, we must have $k$ even $\Rightarrow$ we can not have an odd cycle in $G$.
"$\Leftarrow$:" We can assume w.l.o.g. that $G$ is connected, for otherwise we could treat each of its components separately.
Let $v \in V$ and define
$X = \{x \in V \mid \text{the shortest path from } x \text{ to } v \text{ has even length}\}$,
$Y = V \setminus X$.
It is easy to verify that $G$ is a bipartite graph with partite sets $X$ and $Y$.

# Isomorphic graphs

Note that the following graphs are the same:

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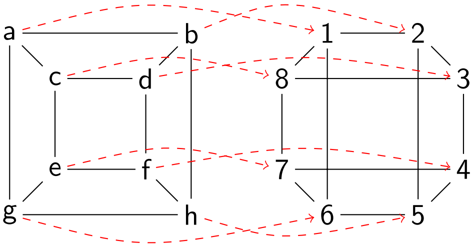


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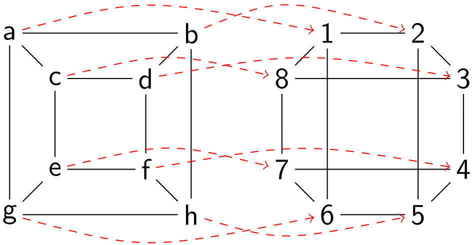


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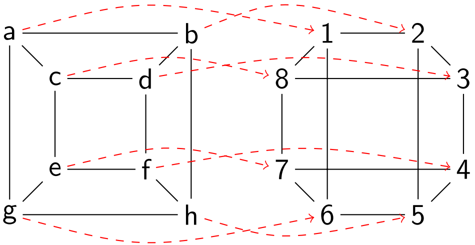


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### Isomorphic graphs

Two graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$ are isomorphic if there is a bijective mapping $f : V_1 \to V_2$ such that $(x, y) \in E_1$ if and only if $(f(x), f(y)) \in E_2$.

# Isomorphic graphs

- When two graphs $G$ and $H$ are isomorphic, it is not uncommon to simply say that "$G = H$" or that "$G$ is $H$."
- If $G$ and $H$ are isomorphic then they have the same order and size. The converse of this statement is not true, as seen in Figure 1 below.

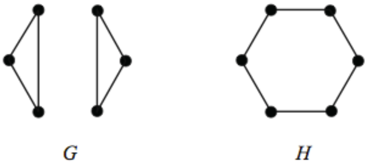


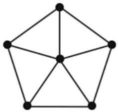

Figure: Two graphs $G$ and $H$ with same order and size, which are not isomorphic.

- If $G$ and $H$ are isomorphic then their degree sequences coincide. The converse of this statement is not true.

# Exercises

1. For $n \geq 2$ prove that $K_n$ has $n(n-1)/2$ edges.
2. Determine whether $K_4$ is a subgraph of $K_{4,4}$. If yes, then exhibit it. If no, then explain why not.
3. The line graph $L(G)$ of a graph $G$ is defined in the following way:
   - the vertices of $L(G)$ are the edges of $G$, $V(L(G)) = E(G)$, and
   - two vertices in $L(G)$ are adjacent if and only if the corresponding edges in $G$ share a vertex.

   a. Find $L(G)$ for the graph

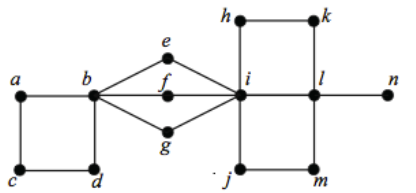
   

   b. Find the complement of $L(K_5)$.

ASSUMPTION: $G = (V, E)$ is a connected graph.

- The distance $d(u, v)$ from node $u$ to node $v$ in $G$ is the length of the shortest path $u - v$ from $u$ to $v$ in $G$.
- The eccentricity $ecc(v)$ of $v$ in $G$ is the greatest distance from $v$ to any other node.

**Example**

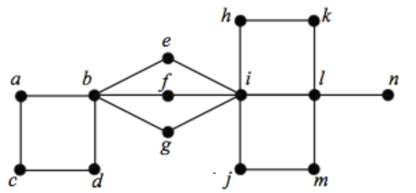



$d(b, k) = 4$, $d(c, m) = 6$.
$ecc(a) = 5$ since the farthest nodes from $a$ are $k, m, n$, and they are a distance 5 from $a$.

ASSUMPTION: $G = (V, E)$ is a connected graph.

- The radius $rad(G)$ of $G$ is the value of the smallest eccentricity.
- The diameter $diam(G)$ of $G$ is the value of the greatest eccentricity.
- The center of $G$ is the set of nodes $v$ such that $ecc(v) = rad(G)$.
- The periphery of $G$ is the set of nodes $v$ such that $ecc(v) = diam(G)$.

## Example




$rad(G) = 3$ and $diam(G) = 6$. The center of $G$ is $\{e, f, g\}$.
The periphery of $G$ is $\{c, k, m, n\}$.

**Theorem 1**

For any connected graph $G$, $rad(G) \leq diam(G) \leq 2\,rad(G)$.

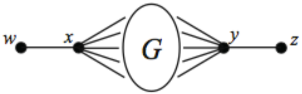
PROOF. By definition, $rad(G) \leq diam(G)$, so we just need to prove the second inequality. Let $u, v$ be nodes in $G$ such that $d(u, v) = diam(G)$. Let $c$ be a node in the center of $G$. Then

$$diam(G) = d(u, v) \leq d(u, c) + d(c, v) \leq 2\,ecc(c) = 2\,rad(G).$$

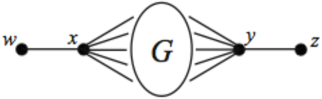
**Theorem 2**

Every graph $G = (V, E)$ is isomorphic to the center of some graph.

PROOF. We construct a new graph $H$ by adding 4 nodes $w, x, y, z$ to $G$ along with the following edges:
$\{(w, x), (y, z)\} \cup \{(x, a) \mid a \in V\} \cup \{(b, y) \mid b \in V\}$.
The newly constructed graph $H$ looks as shown in the figure below.



PROOF OF THEOREM 2 CONTINUED.

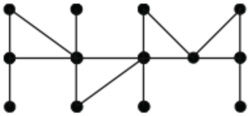



- $ecc(w) = ecc(z) = 4$, $ecc(y) = ecc(x) = 3$, and
- for any node $v$ of $G$: $ecc(v) = 2$.

Therefore $G$ is the center of $H$.

# Exercises

1. Find the radius, diameter and center of the following graph:



2. Find the radius and diameter of each of the following graphs: $P_{2k}$, $P_{2k+1}$, $C_{2k}$, $C_{2k+1}$, $K_n$, and $K_{m,n}$.
3. Given a connected graph $G = (V, E)$ and a positive integer $k$, the $k$-th power of $G$, denoted $G^k$, is the graph whose set of nodes os $V$ and where vertices $u$ and $v$ are adjacent in $G_k$ if and only if $d(u, v) \leq k$ in $G$.
   a. Draw the 2-nd and 3-rd powers of $P_8$ and $C_{10}$.
   b. For a graph $G$ of order $n$, what is $G^{diam(G)}$?

# Trees

A tree is a connected graph which contains no cycles.

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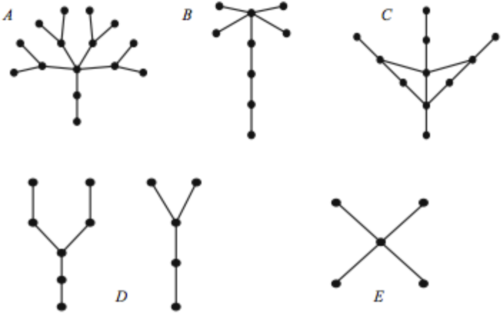
- **Quiz:** Which ones are trees?



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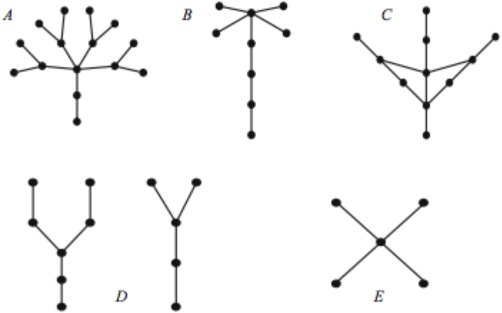


- A forest is a graph whose connected components are trees. E.g., the graph $D$ is a forest.

# Trees

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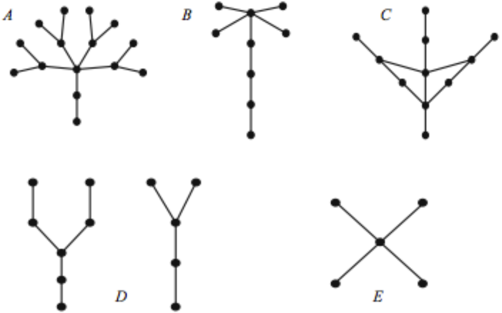
- **Quiz:** Which ones are trees?




- A forest is a graph whose connected components are trees. E.g., the graph $D$ is a forest.
- A leaf in a tree is a node with degree 1.

# Trees

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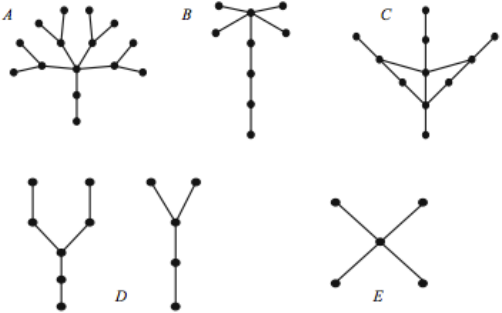


- A forest is a graph whose connected components are trees. E.g., the graph $D$ is a forest.
- A leaf in a tree is a node with degree 1.
- Note that $K_1$ and $K_2$ are the only trees of order 1 and 2, respectively. $P_3$ is the only tree of order 3.

- If $T$ is a tree of order $n$, then $T$ has $n-1$ edges.
- If $F$ is a forest of order $n$ containing $k$ connected components, then $F$ contains $n-k$ edges.
- A graph of order $n$ is a tree if and only if it is connected and contains $n-1$ edges.
- A graph of order $n$ is a tree if and only if it is acyclic and contains $n-1$ edges.
- Let $T$ be the tree of order $n \geq 2$. Then $T$ has at least two leaves.
- In any tree, the center is either a single vertex or a pair of adjacent vertices.

1. Show that every edge in a tree is a bridge.
2. Show that every nonleaf in a tree is a cut vertex.

# References

J. M. Harris, J. L. Hirst, M. J. Mossinghoff. *Combinatorics and Graph Theory, Second Edition.* Springer 2008.

Chapter 1: Graph Theory. Sections §1.1, §1.2 and §1.4.