What is Graph Theory?

- The study of **graphs** as mathematical structures $G = (V, E)$ used to model pairwise relations (*a.k.a.* edges) between objects of a collection $V$.
  - The objects are modeled as **nodes** (or **vertices**) of a set $V$.
  - The pairwise relations are modeled as **edges**, which are elements of a set $E$.

- Graphs differ mainly by the types of edges between nodes. Most common types of graphs are:
  - **Undirected**: there is no distinction between the nodes associated with each edge.
  - **Directed**: edges are arcs from one node to another.
  - **Weighted**: every edge has a weight which is typically a real number.
  - **Labeled**: every edge has its own label.

... Graphs are among the most frequently used models in problem solving.
History of graph theory

- 1736: L. Euler publishes "Seven Bridges of Königsberg" – first paper on graph theory.

Later: Euler’s formula relating the number of edges, vertices, and faces of a convex polyhedron ⇒ generalizations by Cauchy and L’Huillier ⇒ study of topology and special classes of graphs.

1852: De Morgan introduces the “Four Color Map Conjecture”: four is the minimum number of colors required to color any map where bordering regions are colored differently.
  - 1969: Heesch publishes a solving method

1878: The term “graph” was first used by Sylvester in a publication in Nature.

1936: D. König publishes first textbook on graph theory.
**Assumption:** $G = (V, E)$ is a simple graph or digraph.

- The **order** of $G$ is $|V|$, the number of its nodes.
- The **size** of $G$ is $|E|$, the number of its edges.
- The **neighborhood** of $v \in V$ is $N(v) = \{x \in V \mid (v, x) \in E\}$.
- The **closed neighborhood** of $v \in V$ is $N[v] = \{v\} \cup N(v)$.
- The **degree** of $v \in V$ is the number of edges incident with $V$: $\deg(v) = |\{e \in E \mid e = (v, x) \text{ or } e = (x, v) \text{ for some } x \in V\}|$
- The **maximum degree** of $G$ is $\Delta(G) = \max\{\deg(v) \mid v \in V\}$.
- The **minimum degree** of $G$ is $\delta(G) = \min\{\deg(v) \mid v \in V\}$.
- The **degree sequence** of $G$ with order $n$ is the $n$-term sequence (usually written in descending order) of the vertex degrees of its nodes.
G = (V, E) where V = \{a, b, c, d, e, f, g, h\}, E = \\{\{a, d\}, \{a, e\}, \{b, c\}, \{b, e\}, \{b, g\}, \{c, f\}, \{d, f\}, \{d, g\}, \{g, h\}\}

- N(d) = \{a, f, g\}, N[d] = \{a, d, f, g\},
- \Delta(G) = \text{deg}(b) = 3
- \delta(G) = \text{deg}(h) = 1,
- The degree sequence is 3, 3, 3, 2, 2, 2, 2, 1
In a graph $G$, the sum of the degrees of the vertices is equal to twice the number of edges. Consequently, the number of vertices with odd degree is even.

**Combinatorial proof.**
Let $S = \sum_{v \in V} \deg(v)$. Notice that in counting $S$, we count each edge exactly twice. Thus, $S = 2|E|$ (the sum of the degrees is twice the number of edges). Since

$$S = \sum_{\substack{v \in V \\deg(v) \text{ even}}} \deg(v) + \sum_{\substack{v \in V \\deg(v) \text{ odd}}} \deg(v)$$

and $S$ is even, the second sum must be even, thus the number of vertices with odd degree is even.
**Assumption:** $G = (V, E)$ is a simple graph or digraph.

- A **walk** or **path** in $G$ is a sequence of (not necessarily distinct) nodes $v_1, v_2, \ldots, v_k$ such that $(v_i, v_{i+1}) \in E$ for $i = 1, 2, \ldots, k - 1$. Such a walk is sometimes called a $v_1 - v_k$ walk.
  - $v_1$ and $v_k$ are the end vertices of the walk.
  - If the vertices in a walk are distinct, then the walk is called a **simple path**.
  - If the edges in a walk are distinct, then the walk is called a **trail**.

- A **cycle** is a simple path $v_1, \ldots, v_k$ (where $k \geq 3$) together with the edge $(v_k, v_1)$.

- A **circuit** or **closed trail** is a trail that begins and ends at the same node.

- The **length** of a walk (or simple path, trail, cycle, circuit) is its number of edges, counting repetitions.
Introductory concepts
Perambulation and Connectivity

Example

- $a, c, f, c, b, d$ is a walk of length 5.
- $b, a, c, b, d$ is a trail of length 4.
- $d, g, b, a, c, f, e$ is a simple path of length 6.
- $g, d, b, c, a, b, g$ is a circuit.
- $e, d, b, a, c, f, e$ is a cycle.

Note that walks, trails and simple paths can have length 0. The minimum length of a cycle or circuit is 3.
Second Theorem of Graph Theory

**Theorem**

In a graph $G$ with vertices $u$ and $v$, every $u-v$ walk contains a $u-v$ simple path.

**Proof.** Let $W$ be a $u-v$ walk in $G$. We prove this theorem by induction on the length of the walk $W$.

- If $W$ has length 1 or 2, then it is easy to see that $W$ must be a simple path.

- For the induction hypothesis, suppose the result is true for all walks of length $< k$ and suppose $W$ has length $k$. Say that $W$ is $u = w_0, w_1, \ldots, w_{k-1}, w_k = v$. If the nodes are distinct, then $W$ itself is the desired $u-v$ simple path. If not, then let $j$ be the smallest integer such that $w_j = w_r$ for some $r > j$. Let $W_1$ be the walk $u = w_0, \ldots, w_j, w_{r+1} \ldots, w_k = v$. This walk has length strictly less than $k$, and thus $W_1$ contains a $u-v$ simple path by induction hypothesis. Thus $W$ contains a simple $u-v$ path.
Assumptions: $G = (V, E)$ is a simple graph, $v \in V$, $S \subseteq V$, $e \in E$, $T \subseteq E$

- **Vertex deletion:**
  - $G - v$ is the graph obtained by removing $v$ and all edges incident with $v$ from $G$.
  - $G - S$ is the graph obtained by removing each node of $S$ and each edge incident with a node of $S$ from $G$.

- **Edge deletion:**
  - $G - e$ is the graph obtained by removing only the edge $e$ from $G$ (its end nodes stay).
  - $G - T$ is the graph obtained by removing each edge of $T$ from $G$.

- $G$ is connected if every pair of nodes can be joined by a path. Otherwise, $G$ is disconnected.

- A component of $G$ is a maximal connected piece of $G$.

- $v$ is a cut vertex if $G - v$ has more components than $G$.

- $e$ is a bridge if $G - e$ has more components than $G$. 
Operations on graphs and properties related to connectivity

Example (Deletion operations)

\[ G - d \]
\[ G - (c, d) \]
\[ G - \{(e, g), (f, g)\} \]

\(d\) is a cut node in \(G\). \((a, b)\) is a bridge in \(G\).

Example (Connected and disconnected graphs)

\[ G_1 \]
\[ G_2 \]
\[ G_3 \]
**ASSUMPTION:** $G = (V, E)$ is a graph.

- $\emptyset \neq S \subseteq V$ is a **node cut set** of $G$ if $G - S$ is disconnected.
- $G$ is **complete** if every node is adjacent to every other node. We write $K_n$ for the complete graph with $n$ nodes.
  - The complete graphs $K_n$ have no node cut sets because $K_n - S$ is connected for all proper subsets $S$ of the set of nodes.
- If $G$ is not complete then the **connectivity** of $G$, denoted by $\kappa(G)$, is the minimum size of a node cut set of $G$.
  - If $G$ is a connected and incomplete graph of order $n$, then $1 \leq \kappa(G) \leq n - 2$.
  - If $G$ is disconnected, then $\kappa(G) = 0$.
  - If $G = K_n$ then we say that $\kappa(G) = n - 1$.
- If $k > 0$, we say that $G$ is $k$-connected if $k \leq \kappa(G)$. 
Consequences of the definitions

1. A graph is connected if and only if $\kappa(G) \geq 1$.
2. $\kappa(G) \geq 2$ if and only if $G$ is connected and has no cycles.
3. Every 2-connected graph contains at least one cycle.
4. For every graph $G$, $\kappa(G) \leq \delta(G)$.
1. If $G$ is a graph of order $n$, what is the maximum number of edges in $G$?

2. Prove that for any graph $G$ of order at least 2, the degree sequence has at least one pair of repeated entries.

3. Consider the complete graph $K_5$ shown in the following figure.

```
\[ \begin{array}{c}
  a \\
  b \\
  c \\
  d \\
  e \\
\end{array} \]
```

a. How many different simple paths have $c$ as an end vertex?

b. How many different simple paths avoid vertex $c$ altogether?

c. What is the maximum length of a circuit in this graph? Give an example of such a circuit.
4. Let $G$ be a graph where $\delta(G) \geq k$.
   a. Prove that $G$ has a simple path of length at least $k$.
   b. If $k \geq 2$, prove that $G$ has a cycle of length at least $k + 1$.

5. Prove that every closed odd walk in a graph contains an odd cycle.

6. Let $P_1$ and $P_2$ be two paths of maximum length in a connected graph $G$. Prove that $P_1$ and $P_2$ have a common vertex.

7. Prove that every 2-connected graph contains at least one cycle.
1. The complete graphs $K_n$. The graph $K_n$ has order $n$ and a connection between every two nodes. Examples:

2. Empty graphs $E_n$. The graph $E_n$ has order $n$ and no edges. Example:
**Assumption:** \( G = (V, E) \) is a graph.

- The **complement** of \( G \) is the graph \( \overline{G} \) whose node set is the same as that of \( G \) and whose edge set consists of all the edges that are not in \( E \). For example

  \[
  \begin{align*}
  G & \quad \text{and} \quad \overline{G} \\
  \end{align*}
  \]

- \( G \) is **regular** if all its nodes have the same degree. \( G \) is \( r \)-regular if \( \deg(v) = r \) for all nodes \( v \) in \( G \).

  \( K_n \) are \((n-1)\)-regular graphs; \( E_n \) are 0-regular graphs.
Special types of graphs
Cycles, paths, and subgraphs

- The cycle $C_n$ is simply a cycle on $n$ vertices. Example: The graph $C_7$ looks as follows:

- The graph $P_n$ is a simple path on $n$ vertices. For example, the graph $P_6$ looks as follows:

- Given a graph $G = (V, E)$ and a subset $S \subseteq V$, the subgraph of $G$ induced by $S$, denoted $\langle S \rangle_G$, is the subgraph with vertex set $S$ and with edge set $\{(u, v) | u, v \in S \text{ and } (u, v) \in E\}$. So, $\langle S \rangle_G$ contains all vertices of $S$ and all edges of $G$ whose end vertices are both in $S$. 
A graph $G = (V, E)$ is **bipartite** if $V$ can be partitioned into two sets $X$ and $Y$ such that every edge of $G$ has one end vertex in $X$ and the other in $Y$.

- In this case, $X$ and $Y$ are called the **partite sets**.

A **complete bipartite graph** is a bipartite graph with partite sets $X$ and $Y$ such that its edge set is $E = \{ (x, y) \mid x \in X, y \in Y \}$.

- Such a graph is complete bipartite graph denoted by $K_{|X|,|Y|}$. 
Bipartite graphs

The first two graphs in the following figure are bipartite, whereas the third graph is not bipartite.

Complete bipartite graphs

$K_{2,3}$  $K_{1,4}$  $K_{4,4}$
A graph with at least 2 nodes is bipartite if and only if it contains no odd cycles.

Proof.

⇒: Let $G = (V, E)$ be a bipartite graph with partite sets $X$ and $Y$, and let $C = v_1, \ldots, v_k, v_1$ be a cycle in $G$. We can assume $v_1 \in X$ w.l.o.g. Then $v_i \in X$ for all even $i$ and $v_i \in Y$ for all odd $i$. Since $(v_k, v_1) \in E$, we must have $k$ even $\Rightarrow$ we can not have an odd cycle in $G$.

⇐: We can assume w.l.o.g. that $G$ is connected, for otherwise we could treat each of its components separately. Let $v \in V$ and define

$X = \{x \in V \mid$ the shortest path from $x$ to $v$ has even length$\}$,

$Y = V \setminus X$.

It is easy to verify that $G$ is a bipartite graph with partite sets $X$ and $Y$. 
Isomorphic graphs

Note that the following graphs are the same:

This is so because one graph can be redrawn to look like the other.
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\[
\begin{array}{c}
\text{a} \\
\text{b} \\
\text{c} \\
\text{d} \\
\text{e} \\
\text{f} \\
\text{g} \\
\text{h}
\end{array}
\quad
\begin{array}{c}
\text{1} \\
\text{2} \\
\text{8} \\
\text{3} \\
\text{7} \\
\text{4} \\
\text{6} \\
\text{5}
\end{array}
\]

This is so because one graph can be redrawn to look like the other. The idea of isomorphism formalizes this phenomenon.

**Isomorphic graphs**

Two graphs \(G = (V_1, E_1)\) and \(H = (V_2, E_2)\) are **isomorphic** if there is a bijective mapping \(f : V_1 \rightarrow V_2\) such that \((x, y) \in E_1\) if and only if \((f(x), f(y)) \in E_2\).
Isomorphic graphs

- When two graphs $G$ and $H$ are isomorphic, it is not uncommon to simply say that “$G = H$” or that “$G$ is $H$.”
- If $G$ and $H$ are isomorphic then they have the same order and size. The converse of this statement is not true, as seen in Figure 1 below.

![Figure: Two graphs $G$ and $H$ with same order and size, which are not isomorphic.]

- If $G$ and $H$ are isomorphic then their degree sequences coincide. The converse of this statement is not true.
Exercises

1. For \( n \geq 2 \) prove that \( K_n \) has \( n(n - 1)/2 \) edges.

2. Determine whether \( K_4 \) is a subgraph of \( K_{4,4} \). If yes, then exhibit it. If no, then explain why not.

3. The line graph \( L(G) \) of a graph \( G \) is defined in the following way:
   - the vertices of \( L(G) \) are the edges of \( G \), \( V(L(G)) = E(G) \), and
   - two vertices in \( L(G) \) are adjacent if and only if the corresponding edges in \( G \) share a vertex.

   a. Find \( L(G) \) for the graph

   ![Graph](image)

   b. Find the complement of \( L(K_5) \).
**Assumption:** $G = (V, E)$ is a connected graph.

- The distance $d(u, v)$ from node $u$ to node $v$ in $G$ is the length of the shortest path $u - v$ from $u$ to $v$ in $G$.
- The eccentricity $ecc(v)$ of $v$ in $G$ is the greatest distance from $v$ to any other node.

**Example**

$d(b, k) = 4$, $d(c, m) = 6$.

$ecc(a) = 5$ since the farthest nodes from $a$ are $k, m, n$, and they are a distance 5 from $a$. 
**Assumption:** \( G = (V, E) \) is a connected graph.

- The **radius** \( \text{rad}(G) \) of \( G \) is the value of the smallest eccentricity.
- The **diameter** \( \text{diam}(G) \) of \( G \) is the value of the greatest eccentricity.
- The **center** of \( G \) is the set of nodes \( v \) such that \( \text{ecc}(v) = \text{rad}(G) \).
- The **periphery** of \( G \) is the set of nodes \( v \) such that \( \text{ecc}(v) = \text{diam}(G) \).

**Example**

\[ \text{rad}(G) = 3 \text{ and } \text{diam}(G) = 6. \]

The center of \( G \) is \( \{e, f, g\} \).

The periphery of \( G \) is \( \{c, k, m, n\} \).
Distance in graphs
Properties

**Theorem 1**
For any connected graph $G$, $\text{rad}(G) \leq \text{diam}(G) \leq 2 \text{rad}(G)$.

**Proof.** By definition, $\text{rad}(G) \leq \text{diam}(G)$, so we just need to prove the second inequality. Let $u, v$ be nodes in $G$ such that $d(u, v) = \text{diam}(G)$. Let $c$ be a node in the center of $G$. Then

$$\text{diam}(G) = d(u, v) \leq d(u, c) + d(c, v) \leq 2 \text{ecc}(c) = 2 \text{rad}(G).$$

**Theorem 2**
Every graph $G = (V, E)$ is isomorphic to the center of some graph.

**Proof.** We construct a new graph $H$ by adding 4 nodes $w, x, y, z$ to $G$ along with the following edges:

$\{(w, x), (y, z)\} \cup \{(x, a) \mid a \in V\} \cup \{(b, y) \mid b \in V\}$.

The newly constructed graph $H$ looks as shown in the figure below.
Proof of Theorem 2 continued.

- $\text{ecc}(w) = \text{ecc}(z) = 4$, $\text{ecc}(y) = \text{ecc}(x) = 3$, and
- for any node $v$ of $G$: $\text{ecc}(v) = 2$.

Therefore $G$ is the center of $H$. 
1. Find the radius, diameter and center of the following graph:

![Graph Image]

2. Find the radius and diameter of each of the following graphs: $P_{2k}$, $P_{2k+1}$, $C_{2k}$, $C_{2k+1}$, $K_n$, and $K_{m,n}$.

3. Given a connected graph $G = (V, E)$ and a positive integer $k$, the $k$-th power of $G$, denoted $G^k$, is the graph whose set of nodes is $V$ and where vertices $u$ and $v$ are adjacent in $G_k$ if and only if $d(u, v) \leq k$ in $G$.
   
   a. Draw the 2-nd and 3-rd powers of $P_8$ and $C_{10}$.
   b. For a graph $G$ of order $n$, what is $G^{diam(G)}$?
A **tree** is a connected graph which contains no cycles.
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- **Quiz:** Which ones are trees?
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A forest is a graph whose connected components are trees. E.g., the graph $D$ is a forest.
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**Quiz:** Which ones are trees?

- A forest is a graph whose connected components are trees. E.g., the graph $D$ is a forest.
- A leaf in a tree is a node with degree 1.
A **tree** is a connected graph which contains no cycles.

- **Quiz:** Which ones are trees?

- **Forest** is a graph whose connected components are trees. E.g., the graph $D$ is a forest.

- **Leaf** in a tree is a node with degree 1.

- Note that $K_1$ and $K_2$ are the only trees of order 1 and 2, respectively. $P_3$ is the only tree of order 3.
- If $T$ is a tree of order $n$, then $T$ has $n - 1$ edges.
- If $F$ is a forest of order $n$ containing $k$ connected components, then $F$ contains $n - k$ edges.
- A graph of order $n$ is a tree if and only if it is connected and contains $n - 1$ edges.
- A graph of order $n$ is a tree if and only if it is acyclic and contains $n - 1$ edges.
- Let $T$ be the tree of order $n \geq 2$. Then $T$ has at least two leaves.
- In any tree, the center is either a single vertex or a pair of adjacent vertices.
1. Show that every edge in a tree is a bridge.
2. Show that every nonleaf in a tree is a cut vertex.
References


Chapter 1: Graph Theory. Sections §1.1, §1.2 and §1.4.