Lecture 6

Pólya's Enumeration Formula.
Stirling cycle numbers. Stirling set numbers

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Counting in the presence of symmetries

Burnside's Lemma

The number N of equivalence classes of a set of colourings C in the presence of a group of symmetries G is

$$N = \frac{1}{|G|} \sum_{\pi \in G} |C_\pi|$$

where $C_\pi = \{c \in C \mid \pi^*(c) = c\}$ is the invariant set of π in the set of colorings C.

If C is the set of all possible colourings with m colours and π is a cyclic structure made of p cycles, then $|C_\pi| = m^p$.
For instance:

- $|C_{(1,2)(3,4)}| = m^2$
- $|C_{(1)(2)(3)(4)}| = m^4$
- $|C_{(1)(2,4)(3)}| = m^3$

Cycle index of a group

ASSUMPTION: G is a group of n-permutations, and $\pi \in G$

- If π has type $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n]$ then

$$M_\pi = M_\pi(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} x_i^{\lambda_i}$$

  where $x_1, \ldots, x_n$ are unknowns.

- The cycle index of G is

$$P_G(x_1, x_2, \ldots, x_n) = \frac{1}{|G|} \sum_{\pi \in G} M_\pi(x_1, x_2, \ldots, x_n).$$

The dihedral group $G = D_4$ has 8 permutations, and:

$$\begin{aligned}
&M_{(1)(2)(3)(4)} = x_1^4,\\
&M_{(1,3)(2)(4)} = M_{(1)(2,4)(3)} = x_1^2 x_2,\\
&M_{(1,2)(3,4)} = M_{(1,3)(2,4)} = M_{(1,4)(2,3)} = x_2^2,\\
&M_{(1,2,3,4)} = M_{(1,4,3,2)} = x_4.
\end{aligned}$$

If we add these terms and divide the sum by their number, we obtain the cycle index of D_4:

$$P_{D_4}(x_1, x_2, x_3, x_4) = \frac{1}{8}(x_1^4 + 2\,x_1^2 x_2 + 3\,x_2^2 + 2\,x_4),$$

Similarly, for the group C_4 we obtain

$$P_{C_4}(x_1, x_2, x_3, x_4) = \frac{1}{4}(x_1^4 + x_2^2 + 2\,x_4).$$

According to Burnside, the number of colourings of n objects with m colors, by taking into account the symmetries of group G, is $N = P_G(m, m, \ldots, m)$.

Example

The number of 4-beads necklaces with m colors is

$$P_{D_4}(m, m, m, m) = \frac{1}{8}(m^4 + 2\,m^3 + 3\,m^2 + 2\,m).$$

because we already know that

$$P_{D_4}(x_1, x_2, x_3, x_4) = \frac{1}{8}(x_1^4 + 2\,x_1^2 x_2 + 3\,x_2^2 + 2\,x_4)$$

Q: How many 20-beads necklaces can be made by using 3 colors?

A: We compute the cycle index of the symmetry group D_{20}. D_{20} has 20 rotations:

- The rotation with 0° has type $[20, 0, 0, \ldots, 0] \Rightarrow$ monomial x_1^{20}
- 8 rotations with $k \cdot 18^\circ$ where $k \in \{1, 3, 7, 9, 11, 13, 17, 19\}$ have type $[0, \ldots, 0, 1] \Rightarrow$ monomial $8\,x_{20}$
- 4 rotations with $k \cdot 18^\circ$ where $k \in \{2, 6, 14, 18\}$ have type $[\lambda_1, \ldots, \lambda_{20}]$ with $\lambda_{10} = 2$ and $\lambda_j = 0$ for all $j \neq 10 \Rightarrow$ monomial $4\,x_{10}^2$
- 4 rotations with $k \cdot 18^\circ$ where $k \in \{4, 8, 12, 16\}$ have type $[\lambda_1, \ldots, \lambda_{20}]$ with $\lambda_5 = 4$ and $\lambda_j = 0$ for all $j \neq 5 \Rightarrow$ monomial $4\,x_5^4$
- 2 rotations with $k \cdot 18^\circ$ where $k \in \{5, 15\}$ have type $[\lambda_1, \ldots, \lambda_{20}]$ with $\lambda_4 = 5$ and $\lambda_j = 0$ for all $j \neq 4 \Rightarrow$ monomial $2\,x_4^5$
- Rotation with $10 \cdot 18^\circ$ has type $[0, 2, 0, \ldots] \Rightarrow$ monomial x_2^{10}

and 20 reflections

- 10 reflections around axes passing through midpoints of opposite edges of the regular polygon have type $[0, 10, 0, \ldots, 0] \Rightarrow$ monomial $10\,x_2^{10}$
- 10 reflections around axes passing through opposite nodes of the regular polygon have type $[\lambda_1, \ldots, \lambda_{20}]$ with $\lambda_1 = 2$ and $\lambda_9 = 1 \Rightarrow 10\,x_1^2\,x_2^9$

Q: How many 20-beads necklaces can be made by using 3 colors?

A: We compute the cycle index of the symmetry group D_{20}. D_{20} has 20 rotations:

- The rotation with 0° has type $[20, 0, 0, \ldots, 0] \Rightarrow$ monomial x_1^{20}
- 8 rotations with $k \cdot 18^\circ$ where $k \in \{1, 3, 7, 9, 11, 13, 17, 19\}$ have type $[0, \ldots, 0, 1] \Rightarrow$ monomial $8\,x_{20}$
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Q: How many 20-beads necklaces can be made by using 3 colors?

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- The rotation with 0° has type $[20, 0, 0, \ldots, 0] \Rightarrow$ monomial x_1^{20}
- 8 rotations with $k \cdot 18^\circ$ where $k \in \{1, 3, 7, 9, 11, 13, 17, 19\}$ have type $[0, \ldots, 0, 1] \Rightarrow$ monomial $8\, x_{20}$
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▶ $P_{D_{20}}(x_1, x_2, \ldots, x_{20}) = \frac{1}{40}(x_1^{20} + 10\, x_1^2\, x_2^9 + 11\, x_2^{10} + 2\, x_4^5 + 4\, x_5^4 + 4\, x_{10}^2 + 8\, x_{20})$
$\Rightarrow N = P_{20}(3, \ldots, 3) = 87\,230\,157$

The cycle index can be used to solve more complicated problems to count arrangements in the presence of symmetries. For instance:

- How can we find the number of equivalence classes of colourings of *arrangements* of n objects with m colours $y_1, y_2, \ldots, y_m$, if every colour should appear a predefined number of times?

Definition (Pattern Inventory)

The pattern inventory of the colourings of n objects with m colours in the presence of symmetries from a group G is the polynomial

$$F_G(y_1, y_2, \ldots, y_m) = \sum_{\mathbf{v}} a_{\mathbf{v}} y_1^{n_1} y_2^{n_2} \ldots y_m^{n_m}$$

where

- the sum is over all vectors $\mathbf{v} = (n_1, n_2, \ldots, n_m)$ of positive integers such that $n_1 + n_2 + \ldots + n_m = n$, and
- $a_{(n_1, n_2, \ldots, n_m)}$ is the number of non-equivalent colourings of these n objects, where every colour y_i appears exactly n_i times.

Pólya's Enumeration Formula

Example

How many different necklaces can be made with 2 red beads (r), 9 black (b) and 9 white (w)? We assume that the symetries of this necklace are the permutations of the dihedral group D_{20}, made of

- 20 rotations
- 20 symmetries

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- 20 rotations
- 20 symmetries

Answer: This is the coefficient of $r^2b^9w^9$ in the pattern inventory, which is the polynomial

$$F_{D_{20}}(r,b,w) = \sum_{\substack{\mathbf{v}=(i,j,k)\\ i+j+k=20\\ i,j,k\geq 0}} a_{\mathbf{v}} r^i b^j w^k = \sum_{\substack{i+j+k=20\\ i,j,k\geq 0}} a_{(i,j,k)} r^i b^j w^k.$$

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In 1937, G. Pólya found a simple formula to compute the pattern inventory, using the cycle index of the group. (see next slide)

Pólya's Enumeration Formula

Theorem

Suppose S is an arrangement of n objects colorable with m colors $y_1, \ldots, y_m$, and G is a group of n-permutations. Let

$$P_G(x_1, x_2, \ldots, x_n) = \frac{1}{|G|} \sum_{\pi \in G} M_\pi(x_1, x_2, \ldots, x_n)$$

be the cycle index of G. The pattern inventory of all colourings of the objects of S with colours $y_1, \ldots, y_m$ in the presence of symmetries of G is

$$F_G(y_1, \ldots, y_m) = P_G\left(\sum_{i=1}^{m} y_i, \sum_{i=1}^{m} y_i^2, \ldots, \sum_{i=1}^{m} y_i^n\right).$$

The pattern inventory of colourings $F_G(r, g, b)$ with red (r) green (g) and blue (b) of the beads of a necklace with 4 beads (=square vertices) in the presence of symmetries from $G = D_4$ can be computed as follows:

- $m = 3$ because the set of colours is $\{r, g, b\}$
- The cycle index is $P_{D_4}(x_1, x_2, x_3, x_4) = \frac{1}{|D_4|}\sum_{\pi\in D_4} M_\pi(x_1, x_2, x_3, x_4) = \frac{1}{8}(x_1^4 + 2\,x_1^2x_2 + 3\,x_2^2 + 2\,x_4)$

$$
\begin{aligned}
F_G(r,g,b) =& P_{D_4}(r+g+b, r^2+g^2+b^2, r^3+g^3+b^3, r^4+g^4+b^4)\\
=& \frac{1}{8}((r+g+b)^4 + 2\,(r+g+b)^2(r^2+g^2+b^2)\\
&+ 3\,(r^2+g^2+b^2)^2 + 2\,(r^4+g^4+b^4))\\
=& r^4+g^4+b^4+r^3g+r\,g^3+r^3b+r\,b^3+g^3b+g\,b^3\\
&+2\,r^2g^2+2\,r^2b^2+2\,g^2b^2+2\,r^2g\,b+2\,r\,g^2b+2\,r\,g\,b^2
\end{aligned}
$$

E.g., there are 2 colorings with 1 red bead, 1 green, and 2 blue.

The pattern inventory of colourings $F_G(r, g, b)$ with red (r) green (g) and blue (b) of the beads of a necklace with 4 beads (=square vertices) in the presence of symmetries from $G = C_4$ can be computed as follows:

- $m = 3$ because the set of colourings is $\{r, g, b\}$
- The cycle index is
$$P_{C_4}(x_1, x_2, x_3, x_4) = \frac{1}{|C_4|} \sum_{\pi \in C_4} M_\pi(x_1, x_2, x_3, x_4) = \frac{1}{4}(x_1^4 + x_2^2 + 2\,x_4)$$

$$\begin{aligned} F_G(r, g, b) =& P_{C_4}(r + g + b, r^2 + g^2 + b^2, r^3 + g^3 + b^3, r^4 + g^4 + b^4) \\ =& \frac{1}{4}\left((r + g + b)^4 + (r^2 + g^2 + b^2)^2 + 2\,(r^4 + g^4 + b^4)\right) \\ =& r^4 + g^4 + b^4 + r^3 g + r\,g^3 + r^3 b + r\,b^3 + g^3 b + g\,b^3 \\ & + 2\,r^2 g^2 + 2\,r^2 b^2 + 2\,g^2 b^2 + 3\,r^2 g\,b + 3\,r\,g^2 b + 3\,r\,g\,b^2 \end{aligned}$$

E.g., there are 3 colourings with 1 red bead, 1 green, and 2 blue.

Stirling cycle numbers

Problem

In how many ways can n persons be seated at k round tables, such that no table is unoccupied? At every table can stay any number o persons between 1 and n.

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ANSWER: Every answer to this problem is described by a cycle structure with k disjoint structures $C_1 \dots C_k$ where C_i is the cycle describing the people seated at table i.

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Example

The cycle structure $(1, 2, 4)(3, 6, 9, 10)(5)(7, 8)$ represents a possible arrangement of 10 persons at 4 round tables:

- The people at one table are arranged 1,2,4 clockwise.
- The people at another table are arranged 3,6,9,10 clockwise.
- At another table stays only person 5.
- At the remaining table are persons 7 and 8.

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Definition

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- QUESTION: How to compute directly $\left[{n \atop k}\right]$?
- ANSWER: Identify a recursive definition for Stirling cycle numbers, and then solve it.

1. We can not place n persons at 0 tables, unless $n = 0$ (in this special case, the number is assumed to be 1). Thus

$$\begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n > 0. \end{cases}$$

2. $n \geq 1$ persons can be seated at 1 table in $(n-1)!$ ways. Thus:

$$\begin{bmatrix} n \\ 1 \end{bmatrix} = (n-1)! \quad \text{if } n \geq 1.$$

3. n persons can be seated at n tables in just 1 way: every person is alone at a table. Thus: $\begin{bmatrix} n \\ n \end{bmatrix} = 1$.
4. n persons can be seated at $n-1$ tables as follows: all persons, except one couple, stay alone at a table. Thus

$$\begin{bmatrix} n \\ n-1 \end{bmatrix} = \text{number of possible couples} = \binom{n}{2}.$$

5. If the number of tables k is negative or if there are more tables than persons, the problem has no solution. Thus:

$$\begin{bmatrix} n \\ k \end{bmatrix} = 0 \text{ if } k < 0 \text{ or } k > n.$$

6. Every permutation has a cycle structure made of k cycles, where $1 \leq k \leq n$. According to the rule of sum

$$\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} = n!$$

How can we seat $n > 0$ persons at $k > 0$ round tables?

We distinguish two disjoint cases:

1. Place the first $n-1$ persons at $k-1$ round tables, and afterwards place person n at table k. This case can be performed in $\left[{n-1 \atop k-1}\right]$ ways.
2. Place $n-1$ persons at k round tables, and afterwards add person n together with other persons at a round table.
   - Placing $n-1$ persons at k tables can be done in $\left[{n-1 \atop k}\right]$ ways.
   - Placing person n at a round table = placing person n to the left of one of the other persons $i \in \{1, 2, \ldots, n-1\} \Rightarrow n-1$ ways.

   $\Rightarrow$ This case can be performed in $(n-1) \cdot \left[{n-1 \atop k}\right]$ ways.

According to the rule of sum

$$\left[{n \atop k}\right] = (n-1)\left[{n-1 \atop k}\right] + \left[{n-1 \atop k-1}\right] \quad \text{if } n \geq 1 \text{ and } k \geq 1.$$

- We already know that the binomial formula holds $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$. For $y = 1$ we get:

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

  Also, in a previous lecture we gave a combinatorial proof that

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

- We just proved combinatorial proof that

$$\begin{bmatrix} n \\ k \end{bmatrix} = (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$$

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We want to get a formula for Stirling cycle numbers, which is similar to the binomial formula.

Let $G_n(x) = \sum_k \left[{n \atop k} \right] x^k$. Then $G_0(x) = \left[{0 \atop 0} \right] x^0 = 1 \cdot 1 = 1$, and for $n \geq 1$

$$\begin{aligned} G_n(x) &= \sum_k \left[{n \atop k} \right] x^k \\ &= (n-1) \sum_k \left[{n-1 \atop k} \right] x^k + \sum_k \left[{n-1 \atop k-1} \right] x^k \\ &= (n-1) G_{n-1}(x) + x\, G_{n-1}(x) \\ &= (x+n-1) G_{n-1}(x) \end{aligned}$$

$$\Rightarrow G_n(x) = \underbrace{x \cdot (x+1) \cdot (x+2) \cdot \ldots \cdot (x+n-1)}_{\text{notation: } x^{\bar{n}}}.$$

Thus $x^{\bar{n}} = \sum_k \left[{n \atop k} \right] x^k$.

This is an infinite triangle of Stirling cycle numbers growing downwards:

$\begin{bmatrix} n \\ k \end{bmatrix}$	$k=0$	1	2	3	4	5	6	7	8	$n!$
$n=0$	1									1
1	0	1								1
2	0	1	1							2
3	0	2	3	1						6
4	0	6	11	6	1					24
5	0	24	50	35	10	1				120
6	0	120	274	225	85	15	1			720
7	0	720	1764	1624	735	175	21	1		5040
8	0	5040	13068	13132	6769	1960	322	28	1	40320

Recursive formula used in the computation:

$$\begin{bmatrix} n \\ k \end{bmatrix} = (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$$

This is an infinite triangle of binomial numbers growing downwards:

$\binom{n}{k}$	$k=0$	1	2	3	4	5	6	7	8	$n!$
$n=0$	1									1
1	1	1								1
2	1	2	1							2
3	1	3	3	1						6
4	1	4	6	4	1					24
5	1	5	10	10	5	1				120
6	1	6	15	20	15	6	1			720
7	1	7	21	35	35	21	7	1		5040
8	1	8	28	56	70	56	28	8	1	40320

Recursive formula used in the computation:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Stirling set numbers

Problem

In how many ways can we divide n persons in k non-empty and disjoint groups, if the order of persons in one group does not matter?

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In how many ways can we divide n persons in k non-empty and disjoint groups, if the order of persons in one group does not matter?

Example

The set $\{1, 2, 3\}$ can be partitioned in 2 non-empty subsets in 3 ways: $\{1, 2\}, \{3\}$; $\{1, 3\}, \{2\}$; and $\{1\}, \{2, 3\}$.

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In how many ways can we divide n persons in k non-empty and disjoint groups, if the order of persons in one group does not matter?

Example

The set $\{1, 2, 3\}$ can be partitioned in 2 non-empty subsets in 3 ways: $\{1, 2\}, \{3\}$; $\{1, 3\}, \{2\}$; and $\{1\}, \{2, 3\}$.

Definition

The number of ways in which we can partition a set of n elements in exactly k non-empty and disjoint subsets is the Stirling set number $\left\{ {n \atop k} \right\}$.
Often in the literature this number is denoted by $S(n, k)$ instead of $\left\{ {n \atop k} \right\}$.

1. There is only one way to place n people in one group, and also only one way to split n people in n groups. Thus:
$$\left\{ n \atop 1 \right\} = \left\{ n \atop n \right\} = 1.$$
2. We can not place $n > 0$ people in 0 groups. If $n = 0$ then we assume there there is 1 way to place 0 people in 0 groups. Thus:
$$\left\{ n \atop 0 \right\} = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n > 0. \end{cases}$$
3. Splitting n people in $n-1$ groups amounts to choosing a couple of persons for one group; all other persons are alone in their group. Thus
$$\left\{ n \atop n-1 \right\} = \binom{n}{2}.$$
4. It is obvious that
$$\left\{ n \atop k \right\} = 0 \quad \text{if } k < 0 \text{ or } k > n.$$

How can we split $n > 0$ persons in $k > 0$ non-empty and disjoint subsets?

We distinguish 2 disjoint cases:

1. We split the first $n-1$ persons in $k-1$ groups; then person n is obliged to form a singleton group $\{n\} \Rightarrow \left\{ {n-1 \atop k-1} \right\}$ possibilities.
2. We split the first $n-1$ persons in k groups $\Rightarrow \left\{ {n-1 \atop k} \right\}$ possibilities; afterwards, we add person n to one of those k groups$\Rightarrow k \cdot \left\{ {n-1 \atop k} \right\}$ possibilities.

According to the rule of sum

$$\left\{ {n \atop k} \right\} = k \cdot \left\{ {n-1 \atop k} \right\} + \left\{ {n-1 \atop k-1} \right\} \quad \text{if } n \geq 1 \text{ and } k \geq 1.$$

This is an infinite triangle of Stirling set numbers growing downwards:

$\left\{ n \atop k \right\}$	$k = 0$	1	2	3	4	5	6	7	8
$n = 0$	1								
1	0	1							
2	0	1	1						
3	0	1	3	1					
4	0	1	7	6	1				
5	0	1	15	25	10	1			
6	0	1	31	90	65	15	1		
7	0	1	63	301	350	140	21	1	
8	0	1	127	966	1701	1050	266	28	1

Recursive formula used in the computation:

$$\left\{ n \atop k \right\} = k \cdot \left\{ n-1 \atop k \right\} + \left\{ n-1 \atop k-1 \right\}.$$

References

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