Lecture 6
Pólya’s Enumeration Formula.
Stirling cycle numbers. Stirling set numbers

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Counting in the presence of symmetries

**Burnside’s Lemma**

The number $N$ of equivalence classes of a set of colourings $C$ in the presence of a group of symmetries $G$ is

$$N = \frac{1}{|G|} \sum_{\pi \in G} |C_\pi|$$

where $C_\pi = \{ c \in C \mid \pi^*(c) = c \}$ is the invariant set of $\pi$ in the set of colorings $C$.

If $C$ is the set of all possible colourings with $m$ colours and $\pi$ is a cyclic structure made of $p$ cycles, then $|C_\pi| = m^p$.

For instance:

- $|C_{(1,2)(3,4)}| = m^2$
- $|C_{(1)(2)(3)(4)}| = m^4$
- $|C_{(1)(2,4)(3)}| = m^3$
Assumption: \( G \) is a group of \( n \)-permutations, and \( \pi \in G \)

- If \( \pi \) has type \( \lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n] \) then

\[
M_\pi = M_\pi(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} x_i^{\lambda_i}
\]

where \( x_1, \ldots, x_n \) are unknowns.

- The cycle index of \( G \) is

\[
P_G(x_1, x_2, \ldots, x_n) = \frac{1}{|G|} \sum_{\pi \in G} M_\pi(x_1, x_2, \ldots, x_n).
\]
The dihedral group $G = D_4$ has 8 permutations, and:

$$
M_{(1)(2)(3)(4)} = x_1^4,
M_{(1,3)(2)(4)} = M_{(1)(2,4)(3)} = x_1^2x_2,
M_{(1,2)(3,4)} = M_{(1,3)(2,4)} = M_{(1,4)(2,3)} = x_2^2,
M_{(1,2,3,4)} = M_{(1,4,3,2)} = x_4.
$$

If we add these terms and divide the sum by their number, we obtain the cycle index of $D_4$:

$$
P_{D_4}(x_1, x_2, x_3, x_4) = \frac{1}{8}(x_1^4 + 2x_1^2x_2 + 3x_2^2 + 2x_4),
$$

Similarly, for the group $C_4$ we obtain

$$
P_{C_4}(x_1, x_2, x_3, x_4) = \frac{1}{4}(x_1^4 + x_2^2 + 2x_4).$$
According to Burnside, the number of colourings of $n$ objects with $m$ colors, by taking into account the symmetries of group $G$, is

$$N = P_G(m, m, \ldots, m).$$

**Example**

The number of 4-beads necklaces with $m$ colors is

$$P_{D_4}(m, m, m, m) = \frac{1}{8}(m^4 + 2m^3 + 3m^2 + 2m).$$

because we already know that

$$P_{D_4}(x_1, x_2, x_3, x_4) = \frac{1}{8}(x_1^4 + 2x_1^2x_2 + 3x_2^2 + 2x_4)$$
Q: How many 20-beads necklaces can be made by using 3 colors?
A: We compute the cycle index of the symmetry group $D_{20}$. $D_{20}$ has 20 rotations:
- The rotation with $0^\circ$ has type [20, 0, 0, ..., 0] ⇒ monomial $x_1^{20}$
- 8 rotations with $k \cdot 18^\circ$ where $k \in \{1, 3, 7, 9, 11, 13, 17, 19\}$ have type [0, ..., 0, 1] ⇒ monomial $8 \times x_{20}$
- 4 rotations with $k \cdot 18^\circ$ where $k \in \{2, 6, 14, 18\}$ have type $[\lambda_1, \ldots, \lambda_{20}]$ with $\lambda_{10} = 2$ and $\lambda_j = 0$ for all $j \neq 10$ ⇒ monomial $4 \times x_{10}^2$
- 4 rotations with $k \cdot 18^\circ$ where $k \in \{4, 8, 12, 16\}$ have type $[\lambda_1, \ldots, \lambda_{20}]$ with $\lambda_5 = 4$ and $\lambda_j = 0$ for all $j \neq 5$ ⇒ monomial $4 \times x_5^4$
- 2 rotations with $k \cdot 18^\circ$ where $k \in \{5, 15\}$ have type $[\lambda_1, \ldots, \lambda_{20}]$ with $\lambda_4 = 5$ and $\lambda_j = 0$ for all $j \neq 4$ ⇒ monomial $2 \times x_4^5$
- Rotation with $10 \cdot 18^\circ$ has type [0, 2, 0, ...] ⇒ monomial $x_2^{10}$

and 20 reflections
- 10 reflections around axes passing through midpoints of opposite edges of the regular polygon have type [0, 10, 0, ..., 0] ⇒ monomial $10 \times x_2^{10}$
- 10 reflections around axes passing through opposite nodes of the regular polygon have type $[\lambda_1, \ldots, \lambda_{20}]$ with $\lambda_1 = 2$ and $\lambda_9 = 1$ ⇒ $10 \times x_1^2 \times x_2^9$
Q: How many 20-beads necklaces can be made by using 3 colors?
A: We compute the cycle index of the symmetry group $D_{20}$. $D_{20}$ has 20 rotations:

- The rotation with $0^\circ$ has type $[20, 0, 0, \ldots, 0] \Rightarrow$ monomial $x_1^{20}$
- 8 rotations with $k \cdot 18^\circ$ where $k \in \{1, 3, 7, 9, 11, 13, 17, 19\}$ have type $[0, \ldots, 0, 1] \Rightarrow$ monomial $8 \cdot x_{20}$
- 4 rotations with $k \cdot 18^\circ$ where $k \in \{2, 6, 14, 18\}$ have type $[\lambda_1, \ldots, \lambda_{20}]$ with $\lambda_{10} = 2$ and $\lambda_j = 0$ for all $j \neq 10 \Rightarrow$ monomial $4 \cdot x_{10}^2$
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- Rotation with $10 \cdot 18^\circ$ has type $[0, 2, 0, \ldots] \Rightarrow$ monomial $x_{10}^{10}$

and 20 reflections

- 10 reflections around axes passing through midpoints of opposite edges of the regular polygon have type $[0, 10, 0, \ldots, 0] \Rightarrow$ monomial $10 \cdot x_2^{10}$
- 10 reflections around axes passing through opposite nodes of the regular polygon have type $[\lambda_1, \ldots, \lambda_{20}]$ with $\lambda_1 = 2$ and $\lambda_9 = 1 \Rightarrow 10 \cdot x_1^2 \cdot x_2^9$
Q: How many 20-beads necklaces can be made by using 3 colors?

A: We compute the cycle index of the symmetry group $D_{20}$. $D_{20}$ has 20 rotations:

- The rotation with $0^\circ$ has type $[20, 0, 0, \ldots, 0] \Rightarrow$ monomial $x_1^{20}$
- 8 rotations with $k \cdot 18^\circ$ where $k \in \{1, 3, 7, 9, 11, 13, 17, 19\}$ have type $[0, \ldots, 0, 1] \Rightarrow$ monomial $8x_2$
- 4 rotations with $k \cdot 18^\circ$ where $k \in \{2, 6, 14, 18\}$ have type $[\lambda_1, \ldots, \lambda_{20}]$ with $\lambda_{10} = 2$ and $\lambda_j = 0$ for all $j \neq 10 \Rightarrow$ monomial $4x_{10}^2$
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- 2 rotations with $k \cdot 18^\circ$ where $k \in \{5, 15\}$ have type $[\lambda_1, \ldots, \lambda_{20}]$ with $\lambda_4 = 5$ and $\lambda_j = 0$ for all $j \neq 4 \Rightarrow$ monomial $2x_4^5$
- Rotation with $10 \cdot 18^\circ$ has type $[0, 2, 0, \ldots] \Rightarrow$ monomial $x_2^{10}$

and 20 reflections

- 10 reflections around axes passing through midpoints of opposite edges of the regular polygon have type $[0, 10, 0, \ldots, 0] \Rightarrow$ monomial $10x_2^{10}$
- 10 reflections around axes passing through opposite nodes of the regular polygon have type $[\lambda_1, \ldots, \lambda_{20}]$ with $\lambda_1 = 2$ and $\lambda_9 = 1 \Rightarrow 10x_2^2x_9^2$

\[ P_{D_{20}}(x_1, x_2, \ldots, x_{20}) = \frac{1}{40} (x_1^{20} + 10x_1^2x_2^9 + 11x_2^{10} + 2x_4^5 + 4x_5^4 + 4x_9^2 + 8x_{20}) \]

\[ \Rightarrow N = P_{20}(3, \ldots, 3) = 87230157 \]
The cycle index can be used to solve more complicated problems to count arrangements in the presence of symmetries. For instance:

- How can we find the number of equivalence classes of colourings of arrangements of $n$ objects with $m$ colours $y_1, y_2, \ldots, y_m$, if every colour should appear a predefined number of times?

**Definition (Pattern Inventory)**

The pattern inventory of the colourings of $n$ objects with $m$ colours in the presence of symmetries from a group $G$ is the polynomial

$$F_G(y_1, y_2, \ldots, y_m) = \sum_v a_v y_1^{n_1} y_2^{n_2} \cdots y_m^{n_m}$$

where

- the sum is over all vectors $v = (n_1, n_2, \ldots, n_m)$ of positive integers such that $n_1 + n_2 + \ldots + n_m = n$, and
- $a_{(n_1, n_2, \ldots, n_m)}$ is the number of non-equivalent colourings of these $n$ objects, where every colour $y_i$ appears exactly $n_i$ times.
Example

How many different necklaces can be made with 2 red beads ($r$), 9 black ($b$) and 9 white ($w$)? We assume that the symmetries of this necklace are the permutations of the dihedral group $D_{20}$, made of

- 20 rotations
- 20 symmetries
Example

How many different necklaces can be made with 2 red beads ($r$), 9 black ($b$) and 9 white ($w$)? We assume that the symmetries of this necklace are the permutations of the dihedral group $D_{20}$, made of

- 20 rotations
- 20 symmetries

Answer: This is the coefficient of $r^2 b^9 w^9$ in the pattern inventory, which is the polynomial

$$F_{D_{20}}(r, b, w) = \sum_{v=(i,j,k)} a_v r^i b^j w^k = \sum_{i+j+k=20} a_{(i,j,k)} r^i b^j w^k.$$
Example

How many different necklaces can be made with 2 red beads \((r)\), 9 black \((b)\) and 9 white \((w)\)? We assume that the symmetries of this necklace are the permutations of the dihedral group \(D_{20}\), made of

- 20 rotations
- 20 symmetries

**Answer:** This is the coefficient of \(r^2 b^9 w^9\) in the pattern inventory, which is the polynomial

\[
F_{D_{20}}(r, b, w) = \sum_{v=(i, j, k) \text{ s.t. } i+j+k=20, i,j,k \geq 0} a_v r^i b^j w^k = \sum_{i+j+k=20, i,j,k \geq 0} a(i,j,k) r^i b^j w^k.
\]

In 1937, G. Pólya found a simple formula to compute the pattern inventory, using the cycle index of the group. (see next slide)
Pólya’s Enumeration Formula

**Theorem**

Suppose $S$ is an arrangement of $n$ objects colorable with $m$ colors $y_1, \ldots, y_m$, and $G$ is a group of $n$-permutations. Let

$$P_G(x_1, x_2, \ldots, x_n) = \frac{1}{|G|} \sum_{\pi \in G} M_\pi(x_1, x_2, \ldots, x_n)$$

be the cycle index of $G$. The pattern inventory of all colourings of the objects of $S$ with colours $y_1, \ldots, y_m$ in the presence of symmetries of $G$ is

$$F_G(y_1, \ldots, y_m) = P_G \left( \sum_{i=1}^m y_i, \sum_{i=1}^m y_i^2, \ldots, \sum_{i=1}^m y_i^n \right).$$
The pattern inventory of colourings $F_G(r, g, b)$ with red ($r$) green ($g$) and blue ($b$) of the beads of a necklace with 4 beads (≡square vertices) in the presence of symmetries from $G = D_4$ can be computed as follows:

- $m = 3$ because the set of colours is $\{r, g, b\}$

- The cycle index is $P_{D_4}(x_1, x_2, x_3, x_4) = \frac{1}{|D_4|} \sum_{\pi \in D_4} M_\pi(x_1, x_2, x_3, x_4) = \frac{1}{8}(x_1^4 + 2x_1^2x_2 + 3x_2^2 + 2x_4)$

$$F_G(r, g, b) = P_{D_4}(r + g + b, r^2 + g^2 + b^2, r^3 + g^3 + b^3, r^4 + g^4 + b^4)$$

$$= \frac{1}{8}((r + g + b)^4 + 2(r + g + b)^2(r^2 + g^2 + b^2)$$

$$+ 3(r^2 + g^2 + b^2)^2 + 2(r^4 + g^4 + b^4))$$

$$= r^4 + g^4 + b^4 + r^3g + r^3b + r^2g^3 + r^2b^3 + g^3b + g^2b^3 + 2r^2g^2 + 2r^2b^2 + 2g^2b^2 + 2r^2gb + 2rg^2b + 2rgb^2$$

E.g., there are 2 colorings with 1 red bead, 1 green, and 2 blue.
The pattern inventory of colourings $F_G(r, g, b)$ with red ($r$) green ($g$) and blue ($b$) of the beads of a necklace with 4 beads (=square vertices) in the presence of symmetries from $G = C_4$ can be computed as follows:

- $m = 3$ because the set of colourings is $\{r, g, b\}$
- The cycle index is
  \[
P_{C_4}(x_1, x_2, x_3, x_4) = \frac{1}{|C_4|} \sum_{\pi \in C_4} M_\pi(x_1, x_2, x_3, x_4) = \frac{1}{4} \left(x_1^4 + x_2^2 + 2x_4\right)
  \]

\[
F_G(r, g, b) = P_{C_4}(r + g + b, r^2 + g^2 + b^2, r^3 + g^3 + b^3, r^4 + g^4 + b^4)
= \frac{1}{4} \left((r + g + b)^4 + (r^2 + g^2 + b^2)^2 + 2 (r^4 + g^4 + b^4)\right)
= r^4 + g^4 + b^4 + r^3g + rg^3 + b^3 + r^3b + r^2b^2 + 3rb^3 + 3g^3b + g^2b^2 + 3rg + 3g^2b + 3r^2g
\]

E.g., there are 3 colourings with 1 red bead, 1 green, and 2 blue.
Problem
In how many ways can \( n \) persons be seated at \( k \) round tables, such that no table is unoccupied? At every table can stay any number of persons between 1 and \( n \).

Answer:
Every answer to this problem is described by a cycle structure with \( k \) disjoint structures \( C_1, \ldots, C_k \) where \( C_i \) is the cycle describing the people seated at table \( i \).

Example
The cycle structure \( (1, 2, 4)(3, 6, 9, 10)(5)(7, 8) \) represents a possible arrangement of 10 persons at 4 round tables:
The people at one table are arranged 1, 2, 4 clockwise.
The people at another table are arranged 3, 6, 9, 10 clockwise.
At another table stays only person 5.
At the remaining table are persons 7 and 8.
Problem
In how many ways can \( n \) persons be seated at \( k \) round tables, such that no table is unoccupied? At every table can stay any number of persons between 1 and \( n \).

**Answer:** Every answer to this problem is described by a cycle structure with \( k \) disjoint structures \( C_1 \ldots C_k \) where \( C_i \) is the cycle describing the people seated at table \( i \).
Problem

In how many ways can \( n \) persons be seated at \( k \) round tables, such that no table is unoccupied? At every table can stay any number of persons between 1 and \( n \).

Answer: Every answer to this problem is described by a cycle structure with \( k \) disjoint structures \( C_1 \ldots C_k \) where \( C_i \) is the cycle describing the people seated at table \( i \).

Example

The cycle structure \((1, 2, 4)(3, 6, 9, 10)(5)(7, 8)\) represents a possible arrangement of 10 persons at 4 round tables:

- The people at one table are arranged 1,2,4 clockwise.
- The people at another table are arranged 3,6,9,10 clockwise.
- At another table stays only person 5.
- At the remaining table are persons 7 and 8.
The Stirling cycle number $\left[ \begin{array}{c} n \\ k \end{array} \right]$ is the number of possibilities to seat $n$ persons at $k$ identical round tables such that no round table is left unoccupied.

Question: How to compute directly $\left[ \begin{array}{c} n \\ k \end{array} \right]$?

Answer: Identify a recursive definition for Stirling cycle numbers, and then solve it.
The Stirling cycle number $\left[ \begin{array}{c} n \\ k \end{array} \right]$ is the number of possibilities to seat $n$ persons at $k$ identical round tables such that no round table is left unoccupied.

From the previous remark results that $\left[ \begin{array}{c} n \\ k \end{array} \right]$ is the number of $n$-permutations whose cycle structure is made of exactly $k$ cycles.
Definition

The **Stirling cycle number** \[^n_k\] is the number of possibilities to seat \(n\) persons at \(k\) identical round tables such that no round table is left unoccupied.

From the previous remark results that \[^n_k\] is the number of \(n\)-permutations whose cycle structure is made of exactly \(k\) cycles.

**Question:** How to compute directly \[^n_k\]?
The Stirling cycle number \([n \atop k]\) is the number of possibilities to seat \(n\) persons at \(k\) identical round tables such that no round table is left unoccupied.

From the previous remark results that \([n \atop k]\) is the number of \(n\)-permutations whose cycle structure is made of exactly \(k\) cycles.

**Question:** How to compute directly \([n \atop k]\)?

**Answer:** Identify a recursive definition for Stirling cycle numbers, and then solve it.
Stirling cycle numbers

Obvious properties

1. We can not place \( n \) persons at 0 tables, unless \( n = 0 \) (in this special case, the number is assumed to be 1). Thus

\[
\begin{bmatrix}
\begin{array}{c}
\n \\
\end{array}
\end{bmatrix}_{0} = \begin{cases}
1 & \text{if } n = 0, \\
0 & \text{if } n > 0.
\end{cases}
\]

2. \( n \geq 1 \) persons can be seated at 1 table in \((n - 1)!\) ways. Thus:

\[
\begin{bmatrix}
\begin{array}{c}
\n \\
\end{array}
\end{bmatrix}_{1} = (n - 1)! \quad \text{if } n \geq 1.
\]

3. \( n \) persons can be seated at \( n \) tables in just 1 way: every person is alone at a table. Thus: \( \begin{bmatrix}
\begin{array}{c}
\n \\
\end{array}
\end{bmatrix}_{n} = 1 \).

4. \( n \) persons can be seated at \( n - 1 \) tables as follows: all persons, except one couple, stay alone at a table. Thus

\[
\begin{bmatrix}
\begin{array}{c}
\n \\
\end{array}
\end{bmatrix}_{n - 1} = \text{number of possible couples} = \binom{n}{2}.
\]
5. If the number of tables $k$ is negative or if there are more tables than persons, the problem has no solution. Thus:

$$\left[ \begin{array}{c} n \\ k \end{array} \right] = 0 \text{ if } k < 0 \text{ or } k > n.$$ 

6. Every permutation has a cycle structure made of $k$ cycles, where $1 \leq k \leq n$. According to the rule of sum

$$\sum_{k=1}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] = n!$$
How can we seat \( n > 0 \) persons at \( k > 0 \) round tables?

We distinguish two disjoint cases:

1. Place the first \( n - 1 \) persons at \( k - 1 \) round tables, and afterwards place person \( n \) at table \( k \). This case can be performed in \( \binom{n-1}{k-1} \) ways.

2. Place \( n - 1 \) persons at \( k \) round tables, and afterwards add person \( n \) together with other persons at a round table.
   - Placing \( n - 1 \) persons at \( k \) tables can be done in \( \binom{n-1}{k} \) ways.
   - Placing person \( n \) at a round table = placing person \( n \) to the left of one of the other persons \( i \in \{1, 2, \ldots, n-1\} \) \( \Rightarrow \) \( n - 1 \) ways.

\( \Rightarrow \) This case can be performed in \( (n-1) \cdot \binom{n-1}{k} \) ways.

According to the rule of sum

\[
\binom{n}{k} = (n-1)\binom{n-1}{k} + \binom{n-1}{k-1} \quad \text{if} \; n \geq 1 \; \text{and} \; k \geq 1.
\]
We already know that the binomial formula holds
\[(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.\] For \(y = 1\) we get:
\[\frac{2}{2}
\]
\[(x + 1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k\]

Also, in a previous lecture we gave a combinatorial proof that
\[\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.\]

We just proved combinatorial proof that
\[
\begin{bmatrix} n \\ k \end{bmatrix} = (n-1) \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.
\]
We already know that the binomial formula holds
\[(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.\] For \(y = 1\) we get:
\[(x + 1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k\]

Also, in a previous lecture we gave a combinatorial proof that
\[\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.\]

We just proved combinatorial proof that
\[\left[ \begin{array}{c} n \\ k \end{array} \right] = (n - 1) \left[ \begin{array}{c} n-1 \\ k \end{array} \right] + \left[ \begin{array}{c} n-1 \\ k-1 \end{array} \right].\]

We want to get a formula for Stirling cycle numbers, which is similar to the binomial formula.
Stirling cycle numbers
Identifying a generative function

Let $G_n(x) = \sum_k \binom{n}{k} x^k$. Then $G_0(x) = \left[\binom{0}{0}\right] x^0 = 1 \cdot 1 = 1$, and for $n \geq 1$

\[
G_n(x) = \sum_k \binom{n}{k} x^k = (n - 1) \sum_k \binom{n - 1}{k} x^k + \sum_k \binom{n - 1}{k - 1} x^k \\
= (n - 1) G_{n-1}(x) + x G_{n-1}(x) \\
= (x + n - 1) G_{n-1}(x)
\]

$\Rightarrow G_n(x) = x \cdot (x + 1) \cdot (x + 2) \cdot \ldots \cdot (x + n - 1).$

Thus $x^n = \sum_k \binom{n}{k} x^k$. 

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This is an infinite triangle of Stirling cycle numbers growing downwards:

<table>
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<tr>
<th>$\left[ \begin{array}{c} n \ k \end{array} \right]$</th>
<th>$k = 0$</th>
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<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>$n!$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 0$</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
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<td>40320</td>
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</tbody>
</table>

Recursive formula used in the computation:

$$\left[ \begin{array}{c} n \\ k \end{array} \right] = (n-1) \left[ \begin{array}{c} n-1 \\ k \end{array} \right] + \left[ \begin{array}{c} n-1 \\ k-1 \end{array} \right].$$
This is an infinite triangle of binomial numbers growing downwards:

<table>
<thead>
<tr>
<th></th>
<th>$k = 0$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>$n!$</th>
</tr>
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<tbody>
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<td>28</td>
<td>8</td>
<td>1</td>
<td>40320</td>
</tr>
</tbody>
</table>

Recursive formula used in the computation:

\[
\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.
\]
Problem

In how many ways can we divide \( n \) persons in \( k \) non-empty and disjoint groups, if the order of persons in one group does not matter?
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Example

The set \( \{1, 2, 3\} \) can be partitioned in 2 non-empty subsets in 3 ways: \( \{1, 2\}, \{3\} \); \( \{1, 3\}, \{2\} \); and \( \{1\}, \{2, 3\} \).
Problem
In how many ways can we divide \( n \) persons in \( k \) non-empty and disjoint groups, if the order of persons in one group does not matter?

Example
The set \( \{1, 2, 3\} \) can be partitioned in 2 non-empty subsets in 3 ways: \( \{1, 2\}, \{3\} \); \( \{1, 3\}, \{2\} \); and \( \{1\}, \{2, 3\} \).

Definition
The number of ways in which we can partition a set of \( n \) elements in exactly \( k \) non-empty and disjoint subsets is the Stirling set number \( \{n\}_k \). Often in the literature this number is denoted by \( S(n, k) \) instead of \( \{n\}_k \).
1. There is only one way to place $n$ people in one group, and also only one way to split $n$ people in $n$ groups. Thus:

$$\binom{n}{1} = \binom{n}{n} = 1.$$ 

2. We can not place $n > 0$ people in 0 groups. If $n = 0$ then we assume there is 1 way to place 0 people in 0 groups. Thus:

$$\binom{n}{0} = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n > 0. \end{cases}$$

3. Splitting $n$ people in $n-1$ groups amounts to choosing a couple of persons for one group; all other persons are alone in their group. Thus

$$\binom{n}{n-1} = \binom{n}{2}.$$ 

4. It is obvious that

$$\binom{n}{k} = 0 \quad \text{if } k < 0 \text{ or } k > n.$$
How can we split $n > 0$ persons in $k > 0$ non-empty and disjoint subsets?

We distinguish 2 disjoint cases:

1. We split the first $n - 1$ persons in $k - 1$ groups; then person $n$ is obliged to form a singleton group $\{n\} \Rightarrow \left\{ \begin{array}{l}
\end{array} \right.$ possibilities.

2. We split the first $n - 1$ persons in $k$ groups $\Rightarrow \left\{ \begin{array}{l}
\end{array} \right.$ possibilities; afterwards, we add person $n$ to one of those $k$ groups $\Rightarrow k \cdot \left\{ \begin{array}{l}
\end{array} \right.$ possibilities.

According to the rule of sum

$$\left\{ \begin{array}{l}
\end{array} \right. = k \cdot \left\{ \begin{array}{l}
\end{array} \right. + \left\{ \begin{array}{l}
\end{array} \right. \text{ if } n \geq 1 \text{ and } k \geq 1.$$
This is an infinite triangle of Stirling set numbers growing downwards:

<table>
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<tr>
<th>( \binom{n}{k} )</th>
<th>k = 0</th>
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<th>4</th>
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</tbody>
</table>

Recursive formula used in the computation:

\[
\binom{n}{k} = k \cdot \binom{n-1}{k} + \binom{n-1}{k-1}.
\]