

Lecture 5

Pólya's Theory of Counting

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Motivating problems

- 1 How many ways can John, Ken, Jim, Jack, and Rick be seated at a round table?
- 2 How many different necklaces with n beads can be formed using m different kinds of beads?

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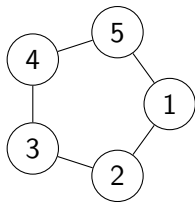
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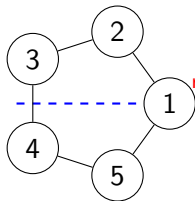
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 - 2 Similarly, two necklaces should be considered identical if we can transform one into the other by rotating the necklace or by turning it over.
- We can rephrase these problems in the language of **group theory**.

Examples of symmetries

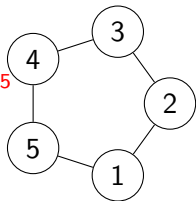
Necklace with 5 beads



↑ symmetry about blue axis
(1)(2,5)(3,4)



rotate with $2\pi/5$
(1, 2, 3, 4, 5)



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- Pólya noticed that the symmetries of an object form a **group**.

Groups

A **group** is a set G together with a binary operator \circ defined on G which satisfies 4 properties:

Closure: $a \circ b \in G$ for all $a, b \in G$.

Associativity: $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a, b, c \in G$

Identity: There exists $e \in G$ such that $e \circ a = a \circ e = a$ for all $a \in G$. The element e is called the **identity** or **neutral element** of G .

Inverses: For every $a \in G$ there exists $b \in G$ such that $a \circ b = b \circ a = e$. The element b is called the **inverse** of a .

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Example

- 1 The set \mathbb{R} of reals with addition $(+)$ is a group:
 - The neutral element is 0, the reverse of $r \in \mathbb{R}$ is $-r$.
- 2 The set S_n of all permutations $\langle a_1, \dots, a_n \rangle$ of $\{1, \dots, n\}$ with function composition \circ is a group.

Subgroups. Permutation groups

- A **subgroup** of a group (G, \circ) with neutral element e is a subset H of G such that
 - $a \circ b \in H$ for all $a, b \in H$ (closure)
 - $e \in H$, and
 - For all $a \in H$ there is $b \in H$ such that $a \circ b = b \circ a = e$.
- A **permutation group** is a subgroup of the set S_n of the permutations of $\{1, \dots, n\}$.
- For a permutation $\pi \in S_n$ we define the powers
 - $\pi^0 = \langle 1, 2, 3, \dots, n \rangle = (1)(2)(3) \dots (n)$,
 - $\pi^1 = \pi$, and
 - $\pi^n = \pi \circ \pi^{n-1}$ if $n > 1$.

Example

If $\pi = (2, 3)(1, 4, 5, 6)$ then

- ▶ $\pi^0 = (1)(2)(3)(4)(5)(6)$, $\pi^1 = \pi = (2, 3)(1, 4, 5, 6)$,
- ▶ $\pi^2 = \pi \circ \pi = (1, 5)(2)(3)(4, 6)$, $\pi^3 = \pi \circ \pi^2 = (1, 6, 5, 4)(2, 3)$,
- ▶ $\pi^4 = \pi \circ \pi^3 = (1)(2)(3)(4)(5)(6) = \pi^0$.

Cyclic groups of permutations. Reflections

For a permutation $\pi \in S_n$, we define the set $\langle \pi \rangle = \{\pi^m \mid m \geq 0\}$. Note that $\langle \pi \rangle$ is a subgroup of S_n . $\langle \pi \rangle$ is the **cyclic subgroup generated by π in S_n** .

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Example

$C_n = \langle (1, 2, \dots, n) \rangle$ is the cyclic group generated by the cycle $(1, 2, \dots, n) \in S_n$. C_n consists of n elements:

$$(1, 2, \dots, n)^0 = \langle 1, 2, \dots, n \rangle = (1)(2) \dots (n)$$

$$(1, 2, \dots, n)^1 = \langle 2, 3, \dots, n, 1 \rangle$$

$$(1, 2, \dots, n)^2 = \langle 3, 4, \dots, 1, 2 \rangle$$

...

$$(1, 2, \dots, n)^{n-1} = \langle n, 1, \dots, n-1 \rangle$$

$$(1, 2, \dots, n)^n = \langle 1, 2, \dots, n \rangle$$

The **reflection** of a permutation $\langle a_1, a_2, \dots, a_n \rangle$ is the permutation $\langle a_n, \dots, a_2, a_1 \rangle$.

C_n is the group of rotational symmetries of a regular polygon with n vertices.

Example

$C_4 = \{(1)(2)(3)(4), (1, 2, 3, 4), (1, 3)(2, 4), (1, 4, 3, 2)\}$ corresponds to the permutation group of the nodes from the figure below, produced by rotations with 90° , 180° , or 270° .

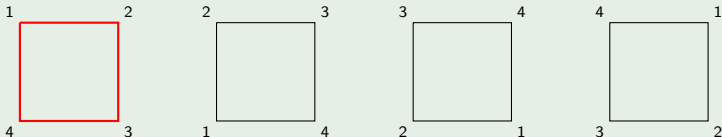


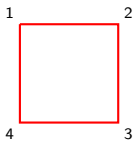
Figure: C_4 as a group of rotations of a square.

- The **dihedral group** D_n consists of the elements of C_n and their reflections. For instance:

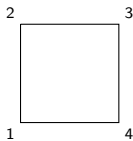
$$\begin{aligned} D_4 &= \{ \langle 1, 2, 3, 4 \rangle, \langle 2, 3, 4, 1 \rangle, \langle 3, 4, 1, 2 \rangle, \langle 4, 1, 2, 3 \rangle \} \cup \\ &\quad \{ \langle 4, 3, 2, 1 \rangle, \langle 1, 4, 3, 2 \rangle, \langle 2, 1, 4, 3 \rangle, \langle 3, 2, 1, 4 \rangle \} \\ &= \{ (1)(2)(3)(4), (1, 2, 3, 4), (1, 3)(2, 4), (1, 4, 3, 2) \} \cup \\ &\quad \{ (1, 4)(2, 3), (1)(2, 4)(3), (1, 2)(3, 4), (1, 3)(2)(4) \}. \end{aligned}$$

- D_n has $2 \cdot n$ elements.
- D_n can be identified with the group of rotational symmetries and reflections of a regular polygon with n elements.

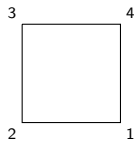
D_4 as group of symmetries of a square



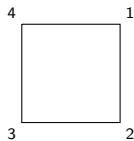
$(1)(2)(3)(4)$



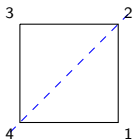
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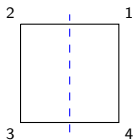
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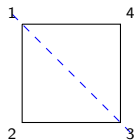
$(1,4,3,2)$



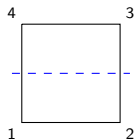
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$(1,4),(2,3)$

Preliminary remarks

- A **transposition** is a cycle of length 2.
- Every cycle (a_1, a_2, \dots, a_p) of an n -permutation can be written as a composition of transpositions:
$$(a_1, a_2, \dots, a_p) = (a_1, a_2)(a_2, a_3) \dots (a_{p-1}, a_p).$$
- Every n -permutation is a composition of cycles (cf. Lecture 4) \Rightarrow every n -permutation is a composition of transpositions.
- An n -permutation is **even** if it is the composition of an even number of transpositions, and **odd** otherwise.
- It can be shown that an n -permutation can not be simultaneously even and odd.

The **alternating group** A_n consists of the even permutations of S_n .

Colorings

A **coloring** of n objects $\{1, 2, \dots, n\}$ is a map $c : \{1, 2, \dots, n\} \rightarrow K$ where $K = \{k_1, \dots, k_m\}$ contains m colors.

- Every coloring c can be represented as a permutation with repetition $\langle c(1), \dots, c(n) \rangle$.
- There are m^n possible colorings.

Example

The colouring $c : \{1, 2, 3, 4\}$ which maps $1 \mapsto r, 2 \mapsto g, 3 \mapsto r, 4 \mapsto r$ is represented by $\langle r, g, r, r \rangle$.

- Let C be the set of all colourings $c : \{1, \dots, n\} \rightarrow K$. If π is a permutation and $c = \langle c(1), \dots, c(n) \rangle$ is a colouring, we define the map $\pi^* : C \rightarrow C$ as follows:
$$\pi^*(\langle c(1), \dots, c(n) \rangle) := \langle c(\pi(1)), \dots, c(\pi(n)) \rangle.$$

Example

If $\pi = (1, 2, 3, 4)$, then $\pi^*(\langle r, g, r, r \rangle) = \langle g, r, r, r \rangle$.

Groups of colourings. Equivalent colourings

G : group of n -permutations

$c_1, c_2 : \{1, 2, \dots, n\} \rightarrow K$ colourings

- c_1 and c_2 are **equivalent** with respect to G , and we write $c_1 \sim_G c_2$ if there is $\pi \in G$ such that $c_2 = \pi^*(c_1)$.

Example

$G = C_4, c = \langle g, g, g, r \rangle$

- $C_4 = \langle \pi \rangle = \{\pi^n \mid n \geq 0\}$ where $\pi = (1, 2, 3, 4)$.
- $C_4 = \{\langle 1, 2, 3, 4 \rangle, \langle 2, 3, 4, 1 \rangle, \langle 3, 4, 1, 2 \rangle, \langle 4, 1, 2, 3 \rangle\}$

\Rightarrow the colourings equivalent with c are:

$$\langle c(1), c(2), c(3), c(4) \rangle = \langle g, g, g, r \rangle$$

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$$\langle c(3), c(4), c(1), c(2) \rangle = \langle g, r, g, g \rangle$$

$$\langle c(4), c(1), c(2), c(3) \rangle = \langle r, g, g, g \rangle$$

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G : group of n -permutations

C : set of colourings $c : \{1, 2, \dots, n\} \rightarrow \{k_1, \dots, k_m\}$ such that $\pi^*(c) \in C$ for every colouring $c \in C$

REMARKS:

- 1 \sim_G is an equivalence relation (reflexive/symmetric/transitive)
 $\Rightarrow C$ can be partitioned in equivalence classes

$$\bar{c} = \{c' \in C \mid c' \sim_G c\} = \{\pi^*(c) \mid \pi \in G\}$$

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 \bar{c} is also called the **orbit** of c under the action of group G .

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How many equivalence classes has C ?

Counting non-equivalent colourings

Concrete example

Example (Square colourings with 2 colors)

$S = \{1, 2, 3, 4\}$ is the set of nodes of a square, and C the set of all possible colourings of the nodes with red (r) and green (g):

$$C = \{\langle g, g, g, g \rangle, \langle g, g, g, r \rangle, \langle g, g, r, g \rangle, \langle g, g, r, r \rangle, \\ \langle g, r, g, g \rangle, \langle g, r, g, r \rangle, \langle g, r, r, g \rangle, \langle g, r, r, r \rangle, \\ \langle r, g, g, g \rangle, \langle r, g, g, r \rangle, \langle r, g, r, g \rangle, \langle r, g, r, r \rangle, \\ \langle r, r, g, g \rangle, \langle r, r, g, r \rangle, \langle r, r, r, g \rangle, \langle r, r, r, r \rangle\}$$

Two colourings are considered equivalent if one can be obtained from the other by rotating the square \Rightarrow we consider $G = C_4 \Rightarrow 6$ equivalence classes of C :

$$\overline{\langle g, g, g, g \rangle} = \{\langle g, g, g, g \rangle\},$$

$$\overline{\langle g, g, g, r \rangle} = \{\langle g, g, g, r \rangle, \langle g, g, r, g \rangle, \langle g, r, g, g \rangle, \langle r, g, g, g \rangle\}$$

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$$\overline{\langle r, r, r, r \rangle} = \{\langle r, r, r, r \rangle\}$$

Counting in the presence of symmetries

Similar problems

- **The round table problem:** S is the set of n places at the table, G is C_n , and C is the collection of $n!$ seating assignments.
- **The necklace problem:** S is the set of n bead positions, G is D_n , and C is the collection of the m^n possible arrangements of the m kinds of beads on the necklace.

Counting in the presence of symmetries

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- **The necklace problem:** S is the set of n bead positions, G is D_n , and C is the collection of the m^n possible arrangements of the m kinds of beads on the necklace.
- We want a general method to solve problems like these.

Counting in the presence of symmetries

Useful notions

G : group of permutations

C : set of colourings $c : \{1, 2, \dots, n\} \rightarrow \{k_1, \dots, k_m\}$ such that $\pi^*(c) \in C$ for all $\pi \in G$

- The **invariant set** of a permutation $\pi \in G$ in C is

$$C_\pi = \{c \in C \mid \pi^*(c) = c\}.$$

- The **stabilizer** of a coloring $c \in C$ is the set of permutations

$$G_c = \{\pi \in G \mid \pi^*(c) = c\}.$$

G_c is always a subgroup of G .

- The set of colorings in C that are equivalent to c under the action of the group G is

$$\bar{c} = \{\pi^*(c) \mid \pi \in G\}.$$

Thus \bar{c} is the equivalence class of c under the relation \sim .

\bar{c} is also called the **orbit** of c under the action of G .

Example (Coloring the nodes of a square with 2 colors)

$$C = \{ \langle g, g, g, g \rangle, \langle g, g, g, r \rangle, \langle g, g, r, g \rangle, \langle g, g, r, r \rangle, \\ \langle g, r, g, g \rangle, \langle g, r, g, r \rangle, \langle g, r, r, g \rangle, \langle g, r, r, r \rangle, \\ \langle r, g, g, g \rangle, \langle r, g, g, r \rangle, \langle r, g, r, g \rangle, \langle r, g, r, r \rangle, \\ \langle r, r, g, g \rangle, \langle r, r, g, r \rangle, \langle r, r, r, g \rangle, \langle r, r, r, r \rangle \}.$$

and the dihedral group $G = D_4$. Then

$$\overline{\langle g, g, g, r \rangle} = \{ \langle g, g, g, r \rangle, \langle g, g, r, g \rangle, \langle g, r, g, g \rangle, \langle r, g, g, g \rangle \}$$

$$G_{\langle g, g, g, r \rangle} = \{ (1)(2)(3)(4), (1, 3)(2)(4) \}$$

$$\overline{\langle g, r, g, r \rangle} = \{ \langle g, r, g, r \rangle, \langle r, g, r, g \rangle \}$$

$$G_{\langle g, r, g, r \rangle} = \{ (1)(2)(3)(4), (1, 3)(2, 4), (1, 3)(2)(4), (1)(2, 4)(3) \}$$

Remark. $|G_{\langle g, g, g, r \rangle}| \cdot |\overline{\langle g, g, g, r \rangle}| = 2 \cdot 4 = 8 = |G|$ and

$$|G_{\langle g, r, g, r \rangle}| \cdot |\overline{\langle g, r, g, r \rangle}| = 4 \cdot 2 = 8 = |G|.$$

Invariant sets, stabilizers, and orbits

Useful properties

Lemma

Suppose a group G acts on a set of colorings C . For any coloring $c \in C$ we have $|G_c| \cdot |\bar{c}| = |G|$.

Burnside's Lemma

The number N of equivalence classes of the set C in the presence of symmetries G is given by

$$N = \frac{1}{|G|} \sum_{\pi \in G} |C_\pi|.$$

Counting in the presence of symmetries

Proof of Burnside's Lemma

$$\begin{aligned}\frac{1}{|G|} \sum_{\pi \in G} |C_{\pi}| &= \frac{1}{|G|} \sum_{\pi \in G} \sum_{c \in C} [\pi^*(c) = c] \\ &= \frac{1}{|G|} \sum_{c \in C} \sum_{\pi \in G} [\pi^*(c) = c] \\ &= \frac{1}{|G|} \sum_{c \in C} |G_c| \\ &= \sum_{c \in C} \frac{1}{\bar{c}} \\ &= \sum_{\bar{c}} \sum_{c \in \bar{c}} \frac{1}{\bar{c}} = \sum_{\bar{c}} 1 = N.\end{aligned}$$

where $[\pi^*(c) = c] := \begin{cases} 1 & \text{if } \pi^*(c) = c \\ 0 & \text{otherwise} \end{cases}$

Burnside's Lemma

Remarks

- To use Burnside's Lemma to count the number of equivalence classes of a set of colorings C , we must compute the size of the invariant set C_π associated with every permutation $\pi \in G$.
- How can we count the size of C_π ?
 - ▶ If c is invariant under the action of π then all objects permuted by the same cycle of π must have the same color.
 - ▶ If π has k disjoint cycles, the number of colorings invariant under the action of π is $|C_\pi| = m^k$, where m is the number of colors.

Example

If S is the set of nodes of a square and $G = D_4$ then

$$|C_{(1,2,3,4)}| = m, \quad |C_{(1,2)(3,4)}| = m^2, \quad |C_{(1,2)(2)(4)}| = m^3, \quad \text{and} \\ |C_{(1)(2)(3)(4)}| = m^4.$$

- 1 J. M. Harris, J. L. Hirst, M. J. Mossinghoff. *Combinatorics and Graph Theory, Second Edition*. Springer 2008.
§2.7. Pólya's Theory of Counting.
- 2 G. Pólya. *Kombinatorische Anzahlbestimmungen für Gruppen, Graphen, und chemische Verbindungen*, Acta Math. 68 (1937), 145–254; English transl. in G. Polya and R. C. Read, *Combinatorial Enumeration of Groups, Graphs, and Chemical Compounds* (1987).