

Lecture 4

The Cycle Structure of Permutations. Advanced Counting Techniques

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Permutations and Cycles

Permutations can be thought as rearrangement operations.

Example

- 1 The permutation $\langle 4, 2, 1, 3 \rangle$ maps 1 to 4, 2 to 2, 3 to 1, and 4 to 3. We can write

$$1 \mapsto 4 \mapsto 3 \mapsto 1, \quad 2 \mapsto 2$$

- 2 The permutation $\langle 2, 1, 3, 5, 7, 4, 6 \rangle$ maps

$$1 \mapsto 2 \mapsto 1, \quad 3 \mapsto 3, \quad 4 \mapsto 5 \mapsto 7 \mapsto 6 \mapsto 4$$

Definition (Cycle)

A **cycle** is a map $\pi : \{v_1, v_2, \dots, v_k\} \rightarrow \{v_1, v_2, \dots, v_k\}$ such that

$$v_1 \mapsto v_2 \mapsto \dots \mapsto v_{k-1} \mapsto v_k \mapsto v_1$$

The mathematical notation of this cycle is $(v_1, \dots, v_k)$.

The cycle (v_1) represents the map $\pi : \{v_1\} \rightarrow \{v_1\}$ with $\pi(v_1) = v_1$.

The cyclic structure of permutations

Remark

Any permutation can be represented as the composition of disjoint cycles. This kind of representation is called the **cyclic structure of a permutation**.

Example

- 1 The permutation $\langle 4, 2, 1, 3 \rangle$ can be represented as a composition of 2 disjoint cycles: $(1, 4, 3)(2)$.
- 2 The permutation $\langle 2, 1, 3, 5, 7, 4, 6 \rangle$ can be represented as a composition of 3 disjoint cycles: $(1, 2)(3)(4, 5, 7, 6)$.

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Properties

The cyclic structure representation of a cycle is not unique: for instance, $(2, 3, 4)$, $(3, 4, 2)$ and $(4, 2, 3)$ are cycles which represent the same function.

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⇒ The cyclic structure of permutations is not unique: for instance, the following cyclic structures represent the same permutation:

- ▷ $(1, 5)(2, 3, 4)$
- ▷ $(1, 5)(3, 4, 2)$
- ▷ $(5, 1)(4, 2, 3)$
- ▷ $(2, 3, 4)(1, 5)$
- ▷ In general, the cyclic structures produced from each other by
 - rotating the cycles of the structure, to left or right, or
 - permuting the cycles within the cycle structurerepresent the same permutation.

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- ▷ In general, the cyclic structures produced from each other by
 - rotating the cycles of the structure, to left or right, or
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- We can define the **canonical cyclic structure** of a permutation as follows:

- ▷ Every cycle is written with smallest element first,
- ▷ Cycles are written in the increasing order of their first element.

Cyclic structures

The construction of the cyclic structure of a permutation

Main idea

- 1 Start computing from 1 the sequence of successors until you reach 1 again. This process builds the first cycle.
- 2 Choose the smallest element not in the first cycle and build the second cycle in the same manner.
- 3 Repeat this process until all elements appear in a cycle.

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Write down the canonical cyclic structures of the following permutations:

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- 2 $\langle 10, 9, 8, 7, 6, 5, 4, 3, 2, 1 \rangle$
 $(1, 10)(2, 9)(3, 8)(4, 7)(5, 6)$

Cyclic structures

Finding the permutation represented by a cyclic structure

Illustrated example

The permutation represented by a cyclic structure

$(1, 3, 4)(2, 6, 7)(5)$ can be found as follow:

- 1 Rotate with 1 to the right all cycles of the initial cyclic structure $\Rightarrow (4, 1, 3)(7, 2, 6)(5)$
- 2 Align the cyclic structure produced before on top of the initial cyclic structure:

$$\begin{array}{ccccccc} (& 4 & , & 1 & , & 3 &) (& 7 & , & 2 & , & 6 &) (& 5 &) \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ (& 1 & , & 3 & , & 4 &) (& 2 & , & 6 & , & 7 &) (& 5 &) \end{array}$$

- 3 Now, we can read off the corresponding permutation:

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \langle & 3 & , & 6 & , & 4 & , & 1 & , & 5 & , & 7 & , & 2 & \rangle \end{array}$$

Cyclic structures

The type of a permutation

The **type** of a permutation π of n elements is the list

$\lambda = [\lambda_1, \dots, \lambda_n]$ where λ_i is the number of cycles of π with length i , for $1 \leq i \leq n$.

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Example

- ① $\langle 1, 2, 3, 4, 5, 6, 7 \rangle = (1)(2)(3)(4)(5)(6)(7)$ has type $[7, 0, 0, 0, 0, 0, 0]$
- ② $\langle 7, 6, 5, 4, 3, 2, 1 \rangle = (1, 7)(2, 6)(3, 5)(4)$ has type $[1, 3, 0, 0, 0, 0, 0]$
- ③ $\langle 1, 3, 2, 6, 7, 8, 9, 4, 10, 5 \rangle = (1)(2, 3)(4, 6, 8)(5, 7, 9, 10)$ has type $[1, 1, 1, 1, 0, 0, 0, 0, 0, 0]$

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- ② $\langle 7, 6, 5, 4, 3, 2, 1 \rangle = (1, 7)(2, 6)(3, 5)(4)$ has type $[1, 3, 0, 0, 0, 0, 0]$
- ③ $\langle 1, 3, 2, 6, 7, 8, 9, 4, 10, 5 \rangle = (1)(2, 3)(4, 6, 8)(5, 7, 9, 10)$ has type $[1, 1, 1, 1, 0, 0, 0, 0, 0, 0]$

REMARK: $[\lambda_1, \dots, \lambda_n]$ is the type of a permutation if and only if

$$1 \cdot \lambda_1 + 2 \cdot \lambda_2 + \dots + n \cdot \lambda_n = n$$

$i \cdot \lambda_i$ = the number of elements in cycles with length i .

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$$\underbrace{c_1^1 \dots c_{\lambda_1}^1}_{\text{cycles with length 1}} \quad \dots \quad \underbrace{c_1^n \dots c_{\lambda_n}^n}_{\text{cycles with length } n}$$

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- We count the cyclic structures for the same permutation
 - ▷ Every cycle c_k^i of length i can be written in i distinct ways $\Rightarrow$ because of this reason, there are $1^{\lambda_1} \cdot 2^{\lambda_2} \cdot \dots \cdot n^{\lambda_n}$ cyclic structures which represent the same permutation
(by Product Rule)
 - ▷ Every permutation of the cycles inside the cyclic structure yields a cyclic structure for the same permutation
 - there are $\lambda_i!$ permutations in every block of cycles of length i
 - $\Rightarrow$ for this reason, here are $\lambda_1! \cdot \lambda_2! \cdot \dots \cdot \lambda_n!$ equivalent cyclic structures.
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$\Rightarrow$ the no. of perms. of type λ is $\frac{n!}{\lambda_1! \cdot \lambda_2! \cdot \dots \cdot \lambda_n! \cdot 1^{\lambda_1} \cdot 2^{\lambda_2} \cdot \dots \cdot n^{\lambda_n}}$

A useful correspondence

Definition

An **integer partition** of a positive integer n is a multiset of strictly positive integers whose sum is n .

Number of integer partition of n = Number of types of n -permutations.

$$[\lambda_1, \dots, \lambda_n] \leftrightarrow \underbrace{\{1, \dots, 1\}}_{\lambda_1 \text{ times}}, \dots, \underbrace{\{n, \dots, n\}}_{\lambda_n \text{ times}}$$

Example (The integer partitions of 5 are the multisets:)

integer partitions	the types
$\{5\}$	$[0, 0, 0, 0, 1]$
$\{4, 1\}$	$[1, 0, 0, 1, 0]$
$\{3, 2\}$	$[0, 1, 1, 0, 0]$
$\{3, 1, 1\}$	$[2, 0, 1, 0, 0]$
$\{2, 2, 1\}$	$[1, 2, 0, 0, 0]$
$\{2, 1, 1, 1\}$	$[3, 1, 0, 0, 0]$
$\{1, 1, 1, 1, 1\}$	$[5, 0, 0, 0, 0]$

- ① Given the permutation $\langle 2, 3, 4, 1, 5, 6 \rangle$
 - ① Which is the cyclic structure of the permutation?
 - ② Which is the canonical cyclic structure of the permutation?
 - ③ Find the type of the permutation!
 - ④ How many permutations have the same type as the given permutation?
- ② List all the integer partitions of 4 and their corresponding types.

Part 2: Advanced counting techniques

Preliminary remarks

- Many interesting counting problems cannot be solved with the counting techniques presented so far.
- Examples:
 - 1 How many n -bit strings don't have two consecutive zeroes?
 - 2 How many ways are there to assign 7 jobs to 3 employees so that each employee is assigned at least one job?

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- Solving linear recurrence relations
- Divide-and-conquer algorithms

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- A **recurrence relation** for a sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms $a_0, a_1, \dots, a_{n-1}$ of the sequence, for all $n \geq n_0$, where $n_0 \geq 0$.

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We will develop techniques to solve various kinds of recurrence relations.

Recurrence relations

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- $a_0 = 3, a_1 = 5, a_n = a_{n-1} - a_{n-2}$ for $n \geq 2$.

All elements of $\{a_n\}$ can be computed recursively:

$$a_2 = a_1 - a_0 = 5 - 3 = 2$$

$$a_3 = a_2 - a_1 = 2 - 5 = -3$$

...

Can we find a general formula to compute a_n directly, as a function of n ?

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- $a_0 = 0, a_1 = 3, a_n = 2 \cdot a_{n-1} - a_{n-2}$ for $n \geq 2$. All elements of $\{a_n\}$ can be computed recursively:

$$a_2 = 2 a_1 - a_0 = 6$$

$$a_3 = 2 a_2 - a_1 = 9$$

...

It can be shown by induction on n that $a_n = 3n$ for all $n \geq 0$.

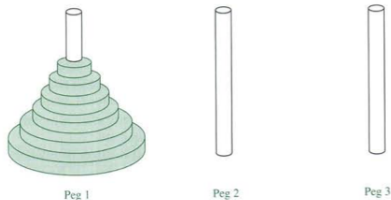
Example: Rabbits and Fibonacci numbers

A young pair of rabbits starts breeding when they are 2 months old, by giving birth to another pair each month. Suppose a zero-months old pair of rabbits is placed on an island. Find a recurrence relation for the number of pairs of rabbits on the island after n months.

Reproducing pairs (at least two months old)	Young pairs (less than two months old)	Month	Reproducing pairs	Young pairs	Total pairs
		1	0	1	1
		2	0	1	1
		3	1	1	2
		4	1	2	3
		5	2	3	5
		6	3	5	8

$$f_1 = 1, f_2 = 1, f_n = f_{n-1} + f_{n-2} \text{ if } n \geq 2.$$

Example: Tower of Hanoi



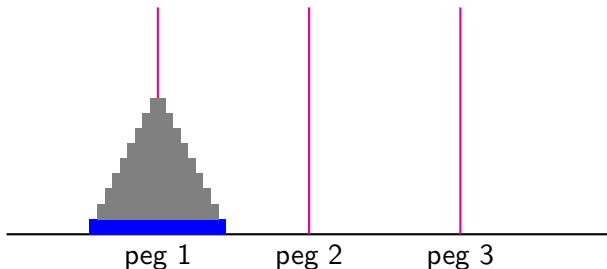
- Move all disks on the second peg in order of size, with the largest disk on the bottom.
- Disks are moved one at a time from one peg to another peg as long as a disk is never placed on top of a smaller disk.

Question: What is the minimum number of moves needed to solve the Tower of Hanoi problem with n disks?

Example: Tower of Hanoi (continued)

A: Let H_n be the minimum number of moves needed to move n disks in order of size, from one peg to another.

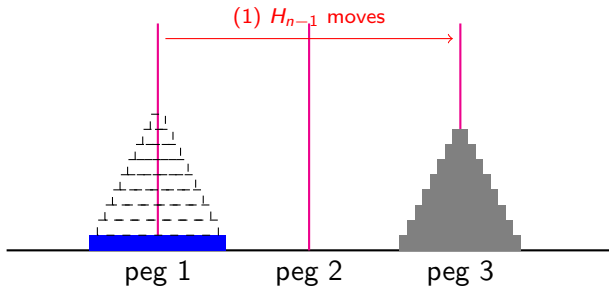
- To place the largest disk on bottom of peg 2, first we must move the $n - 1$ smaller disks from peg 1 to peg 3. The minimum number of moves to do so is H_{n-1} .
- After moving the largest disk from peg 1 to peg 2, we need minimum H_{n-1} moves to move the disks from peg 3 to peg 2.



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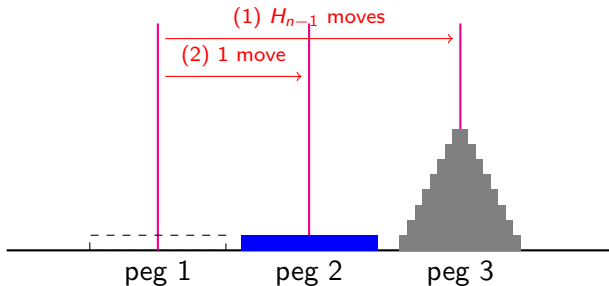
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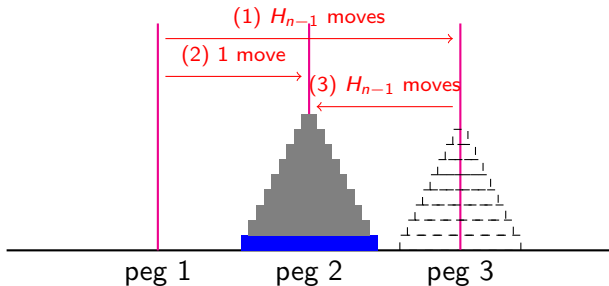
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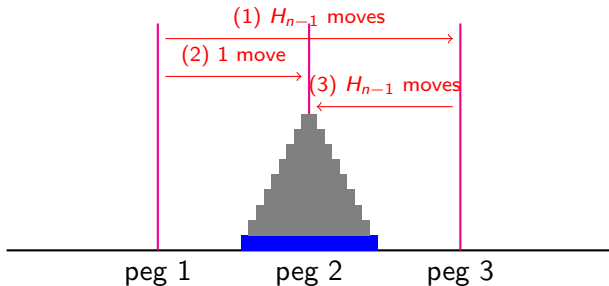
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$$\Rightarrow H_n = H_{n-1} + 1 + H_{n-1} = 2H_{n-1} + 1. \text{ Note that } H_1 = 1.$$

Example: Tower of Hanoi (continued)

- We can use an iterative approach to find the formula for H_n when $n > 1$:

$$\begin{aligned}H_n &= 2 H_{n-1} + 1 \\&= 2(2 H_{n-2} + 1) + 1 = 2^2 H_{n-2} + 2 + 1 \\&= 2^2(2 H_{n-3} + 1) + 2 + 1 = 2^3 H_{n-3} + 2^2 + 2 + 1 \\&\vdots \\&= 2^{n-1} H_1 + 2^{n-2} + \dots + 2 + 1 \\&= 2^{n-1} + 2^{n-2} + \dots + 2 + 1 \\&= \frac{2^n - 1}{2 - 1} = 2^n - 1.\end{aligned}$$

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The myth of the puzzle:

- There are 64 disks, and moving 1 disk takes 1 second
- Minimum time to move the Tower of Hanoi=
 $(2^{64} - 1) s = 18446744073709551615 s \approx 500$ billion years.

Example: Special bit strings

- Find a recurrence relation and give initial conditions for the number of bit strings of length n that do not have two consecutive zeros. How many such bit strings of length 5 do we have?

A: There are 2 disjoint counting tasks:

- Count the n -bit strings with no 2 consec. 0s that end with 1:
- Count the n bit-strings with no 2 consec. 0s that end with 0:

Number of bit strings of length n with no two consecutive 0s:

End with a 1:	Any bit string of length $n - 1$ with no 2 consecutive 0s	1	a_{n-1}
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End with a 0:	Any bit string of length $n - 2$ with no 2 consecutive 0s	1 0	a_{n-2}
---------------	---	-----	-----------

Total: $a_n = a_{n-1} + a_{n-2}$

The bit strings of length 1 are 0 and 1 $\Rightarrow a_1 = 2$, and the bit strings of length 2 without consecutive 0s are 01, 10, 11 $\Rightarrow a_2 = 3$.

Example: Special bit strings (continued)

The number a_n of bit strings of length n without two consecutive zeros is given by the recurrence relation

$$a_1 = 2, \quad a_2 = 3, \quad a_n = a_{n-1} + a_{n-2} \quad \text{if } n \geq 2.$$

$$\Rightarrow a_3 = a_1 + a_2 = 2 + 3 = 5$$

$$\Rightarrow a_4 = a_2 + a_3 = 3 + 5 = 8$$

$$\Rightarrow a_5 = a_3 + a_4 = 5 + 8 = 13.$$

Can we find a general formula to compute a_n directly, as a function of n ?

Linear recurrence relations

- A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k},$$

where $c_1, c_2, \dots, c_k \in \mathbb{R}$ and $c_k \neq 0$.

If we know the k initial conditions

- $a_0 = C_0, a_1 = C_1, \dots, a_{k-1} = C_{k-1},$

then we can compute a_n recursively, for all $n \geq k$.

Example (Linear recurrence relations)

- $\{f_n\}$ where $f_0 = f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ if $n > 1$.
- $\{P_n\}$ where $P_0 = 1$, and $P_n = 1.11 P_{n-1}$ if $n > 0$.

Example (Nonlinear recurrence relations)

$$a_0 = 1, a_1 = 1, a_n = a_{n-1}^2 + a_{n-2} \text{ for all } n \geq 2.$$

Linear recurrence relations

- They occur often in modeling of problems.
- We can find a formula to compute a_n directly from n .

Theorem 1

Consider the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}, \quad a_0 = C_0, \dots, a_{k-1} = C_{k-1}. \quad (1)$$

Suppose $r_1, \dots, r_t$ are the distinct roots of $r^k - c_1 r^{k-1} - \dots - c_k = 0$ with multiplicities $m_1, \dots, m_t$, respectively, so that $m_i \geq 1$ for $i = 1, 2, \dots, t$ and $m_1 + m_2 + \dots + m_t = k$. Then a sequence $\{a_n\}$ is a solution of (1) if and only if

$$\begin{aligned} a_n = & (\alpha_{1,0} + \alpha_{1,1}n + \dots + \alpha_{1,m_1-1}n^{m_1-1})r_1^n \\ & + (\alpha_{2,0} + \alpha_{2,1}n + \dots + \alpha_{2,m_2-1}n^{m_2-1})r_2^n \\ & + \dots + (\alpha_{t,0} + \alpha_{t,1}n + \dots + \alpha_{t,m_t-1}n^{m_t-1})r_t^n \end{aligned}$$

for $n \in \mathbb{N}$, where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j < m_i$.

Linear recurrence relations

Examples

- Find the solution to the recurrence relation

$$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$$

with initial conditions $a_0 = 1$, $a_1 = -2$, and $a_2 = -1$.

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- ANSWER. The characteristic equation of the recurrence relation is $r^3 + 3r^2 + 3r + 1 = 0$, which has a single root $r = -1$ of multiplicity 3 of the characteristic equation.

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$$a_n = \alpha_{1,0}(-1)^n + \alpha_{1,1}n(-1)^n + \alpha_{1,2}n^2(-1)^n.$$

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To find the constants $\alpha_{1,0}, \alpha_{1,1}, \alpha_{1,2}$, use the initial conditions:

$$\begin{cases} a_0 = 1 = \alpha_{1,0} \\ a_1 = -2 = -\alpha_{1,0} - \alpha_{1,1} - \alpha_{1,2} \\ a_2 = -1 = \alpha_{1,0} + 2\alpha_{1,1} + 4\alpha_{1,2} \end{cases} \Rightarrow \begin{cases} \alpha_{1,0} = 1 \\ \alpha_{1,1} = 3 \\ \alpha_{1,2} = -2. \end{cases}$$

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$$\Rightarrow a_n = (1 + 3n - 2n^2)(-1)^n.$$

Nonhomogeneous Recurrences with Constant Coefficients

Definition

A **linear non homogeneous recurrence relation with constant coefficients** is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + F(n),$$

where $c_1, \dots, c_k \in \mathbb{R}$ and $F(n)$ is a function non identically zero depending only on n . The recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$$

is called the **associated recurrence relation**.

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EXAMPLES

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The associated homogeneous relation is $a_n = a_{n-1}$.

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EXAMPLES

- ① $a_n = a_{n-1} + 2^n$ is a non homogeneous recurrence relation. The associated homogeneous relation is $a_n = a_{n-1}$.
- ② $a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$ is a non homogeneous recurrence relation. The associated homogeneous relation is $a_n = a_{n-1} + a_{n-2}$.

Theorem 2

If $\{a_n^{(p)}\}$ is a particular solution of

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + F(n),$$

then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$, where $\{a_n^{(h)}\}$ is a solution of the associated homogeneous recurrence relation

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Q: How can we find a particular solution $\{a_n^{(p)}\}$?

Nonhomogeneous Recurrences with Constant Coefficients

Finding a particular solution

Theorem 3

If $F(n) = (b_t n^t + b_{t-1} n^{t-1} + \dots + b_1 n + b_0) s^n$ with $b_0, \dots, b_{t-1}, b_t, s \in \mathbb{R}$ then

- 1 If s is not a root of the characteristic equation of the associated linear homogeneous recurrence relation, there is a particular solution of the form

$$(p_t n^t + p_{t-1} n^{t-1} + \dots + p_1 n + p_0) s^n.$$

- 2 If s is a root with multiplicity m of the characteristic equation of the associated linear homogeneous recurrence relation, there is a particular solution of the form

$$n^m (p_t n^t + p_{t-1} n^{t-1} + \dots + p_1 n + p_0) s^n.$$

Nonhomogeneous Recurrences with Constant Coefficients

Example 1

Q: What is the form of the solution of the nonlinear recursive relation

$$a_n = 6 a_{n-1} - 9 a_{n-2} + F(n)$$

when $F(n) = n^2 2^n$?

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- $F(n)$ is of the form $Q(n) s^n$ where $Q(n)$ is polynomial of degree $t = 2$, and $s = 2$ is not root of the characteristic equation of the associated homogeneous recurrence $\Rightarrow$ by [Theorem 3](#), a particular solution is of the form $a_n^{(p)} = (p_2 n^2 + p_1 n + p_0) 2^n$.

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- From $a_n^{(p)} = 6 a_{n-1}^{(p)} - 9 a_{n-2}^{(p)} + n^2 2^n$ we obtain
$$2^{n-2}((p_2 - 4)n^2 + (p_1 - 12p_2)n + p_0 - 6p_1 + 24p_2) = 0$$
$$\Rightarrow p_0 = 192, p_1 = 48, p_2 = 4$$

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$$\Rightarrow a_n = a_n^{(p)} + a_n^{(h)} = (4n^2 + 48n + 192) 2^n + (b_1 n + b_0) 3^n.$$

Nonhomogeneous Recurrences with Constant Coefficients

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Q: What is the form of the solution of the nonlinear recursive relation

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A: The associated linear homogeneous recurrence relation for a_n is $a_n = a_{n-1}$. It has the characteristic equation $r - 1 = 0$, thus the solution of it is of the form

$$a_n^{(h)} = c \cdot 1^n = c \text{ where } c \in \mathbb{R}.$$

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- To find the values of p_0 and p_1 , we know that $a_n^{(p)} = a_{n-1}^{(p)} + n$, which implies $n(2p_1 - 1) + (p_0 - p_1) = 0$, which means that $p_0 = p_1 = \frac{1}{2}$. Hence $a_n^{(p)} = \frac{n^2}{2} + \frac{n}{2} = \frac{n(n+1)}{2}$.

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- By [Theorem 2](#), we have $a_n = a_n^{(p)} + a_n^{(h)} = c + \frac{n(n+1)}{2}$. Also, we have $1 = a_1 = c + \frac{1 \cdot 2}{2} = c + 1$, so $c = 0$. Thus $a_n = \frac{n(n+1)}{2}$.

Divide-and-Conquer algorithms and recurrences

How do they work?

- Divide** a problem into one or more instances of the same problem, but of smaller size.
- Conquer** the problem by using the solutions of the smaller problems to find a solution of the original problem.

Divide-and-Conquer algorithms and recurrences

How do they work?

Divide a problem into one or more instances of the same problem, but of smaller size.

Conquer the problem by using the solutions of the smaller problems to find a solution of the original problem.

Typical examples:

- ① Binary search for an element in a sorted list.
- ② Sorting a list by successively splitting the list into halves, and sort each half separately.
- ③ ...

Divide-and-Conquer recurrence relations

Phases of a divide-and-conquer algorithm

- Divide a problem of size n into b subproblems of size n/b .
 - REMARK. In reality, not all subproblems have exactly the same size: some have size $\lceil n/b \rceil$, other have size $\lfloor n/b \rfloor$.
- ASSUMPTIONS
 - $f(n/b) :=$ number of operations required to solve problems of size n/b
 - $a :=$ number of subproblems that have to be solved.
 - $g(n) :=$ number of extra operations required to combine the solutions of subproblems into a solution of the initial problem (the conquer step)

$$\Rightarrow f(n) = a f(n/b) + g(n).$$

This is called a **divide-and-conquer recurrence relation**.

Divide-and-Conquer

Example: Binary Search

Search an item in a sorted sequence of n items, as follows:

- **Split** the initial sorted sequence into 2 sorted sequences of size $n/2$, and **choose** the subsequence in which to search further
 $\Rightarrow$ **one** subproblem of size $n/2$,
- 2 comparisons are needed to determine:
 - ① which half of the sequence to use, and
 - ② if there are any elements in the list.
- $\Rightarrow$ divide-and-conquer relation

$$f(n) = f(n/2) + 2.$$

Divide-and-Conquer

Example: MERGESORT

```
procedure MERGESORT( $L = a_1, \dots, a_n$ )  
if  $n > 1$  then  
     $m = \lfloor n/2 \rfloor$   
     $L_1 = a_1, \dots, a_m$   
     $L_2 = a_{m+1}, \dots, a_n$   
     $L := \text{merge}(\text{MERGESORT}(L_1), \text{MERGESORT}(L_2))$   
/*  $L$  is now sorted into elements in nondecreasing order */
```

```
procedure MERGE( $L_1, L_2$ : sorted list)  
 $L := \text{empty list}$   
while  $L_1$  and  $L_2$  are both non-empty  
    remove smaller of first element of  $L_1$  and  $L_2$  from the list it is in,  
    and put it at the right end of  $L$ .  
    if removal of this element makes one list empty  
    then remove all elements from the other list and append them to  $L$ .
```

Divide-and-Conquer

Example: Merge-sort (continued)

Merging the sorted lists 2,3,5,6 and 1,4.			
First list	Second list	Merged list	Comparison
2,3,5,6	1,4		$1 < 2$
2,3,5,6	4	1	$2 < 4$
3,5,6	4	1,2	$3 < 4$
5,6	4	1,2,3	$4 < 5$
5,6		1,2,3,4	
		1,2,3,4,5,6	

Divide-and-Conquer

Example: Merge-sort (continued)

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		1,2,3,4,5,6	

REMARKS

- 1 MERGESORT uses fewer than n comparisons to merge 2 lists with $n/2$ elements each.
- 2 The number of comparisons used by MERGESORT to sort a list of n elements is less than $M(n)$, where

$$M(n) = 2 M(n/2) + n.$$

Divide-and-Conquer relations

Estimating the size of solutions

Theorem 4

Let f be an increasing function that satisfies the recurrence relation

$$f(n) = af(n/b) + c$$

whenever n is divisible by b , where $a \geq 1$, b is an integer greater than 1, and $c \in \mathbb{R}$ is positive. Then

$$f(n) \text{ is } \begin{cases} O(n^{\log_b(a)}) & \text{if } a > 1 \\ O(\log n) & \text{if } a = 1. \end{cases}$$

Furthermore, when $n = b^k$, where k is a positive integer, then

$$f(n) = C_1 n^{\log_b a} + C_2,$$

where $C_1 = f(1) + C/(a - 1)$ and $C_2 = -c/(a - 1)$.

Divide-and-Conquer relations

Estimating the size of solutions

Theorem 5 (Master Theorem)

Let f be an increasing function that satisfies the recurrence relation

$$f(n) = a f(n/b) + c n^d$$

whenever $n = b^k$, where k is a positive integer, $a \geq 1$, b is an integer greater than 1, and $c, d \in \mathbb{R}$ with $c > 0$ and $d \geq 0$. Then

$$f(n) \text{ is } \begin{cases} O(n^d) & \text{if } a < b^d, \\ O(n^d \log n) & \text{if } a = b^d, \\ O(n^{\log_b a}) & \text{if } a > b^d. \end{cases}$$

Divide-and-Conquer relations

Estimating the size of solutions

Theorem 5 (Master Theorem)

Let f be an increasing function that satisfies the recurrence relation

$$f(n) = a f(n/b) + c n^d$$

whenever $n = b^k$, where k is a positive integer, $a \geq 1$, b is an integer greater than 1, and $c, d \in \mathbb{R}$ with $c > 0$ and $d \geq 0$. Then

$$f(n) \text{ is } \begin{cases} O(n^d) & \text{if } a < b^d, \\ O(n^d \log n) & \text{if } a = b^d, \\ O(n^{\log_b a}) & \text{if } a > b^d. \end{cases}$$

Example (Complexity of MERGESORT)

$M(n) = a M(n/b) + c n^d$ where $a = b = 2$, $c = d = 1$

$\Rightarrow M(n)$ is $O(n \log n)$.

Section 3.1 from

- S. Pemmaraju, S. Skiena. *Combinatorics and Graph Theory with Mathematica*. Cambridge University Press. 2003.

Chapter 7 of

- Kenneth H. Rosen. *Discrete Mathematics and Its Applications*. Sixth Edition. McGraw-Hill, 2007.