Lecture 3

Permutations with repetition. Combinations. Enumeration, ranking and unranking algorithms

Isabela Drămnesc UVT

Computer Science Department,
West University of Timişoara,
Romania

Outline

- Enumeration, ranking and unranking algorithms for permutations with repetition
- Binary represention of subsets
  - ▷ Ranking and unranking algorithms
- Fast generation of all subsets
  - ▷ Gray codes; properties
- Lexicographically ordered combinations (or subsets)
- *r*-combinations: ranking and unranking algorithms

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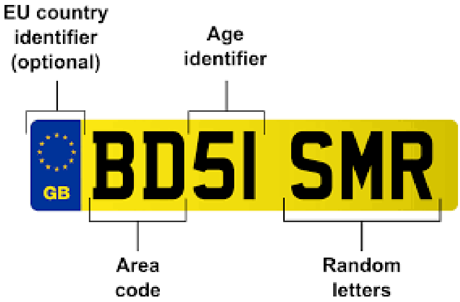
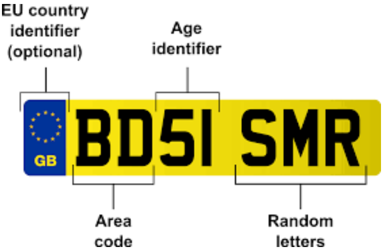
In U.K. the license plates are made up of

- the regional flag followed by
- a two-digit local area code,
- a two-digit age identifier (corresponding to the year the vehicle is registered), followed by
- a three-digit sequence of letters.



How many licence plates are possible in U.K.?

Questions! Ideas?



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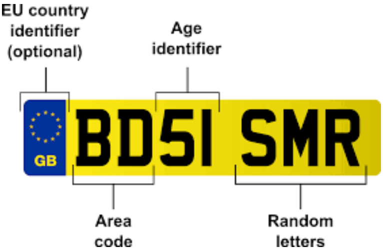

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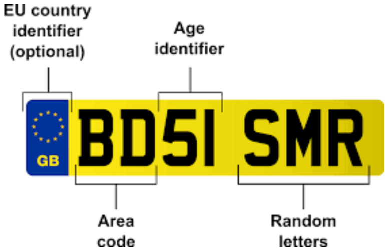


1. It is a Permutation problem or a Combination problem? Explain;
2. It is a Permutation with Repetitions problem or without repetitions? Explain.
3. How to compute Permutations with repetitions? Where to start from?

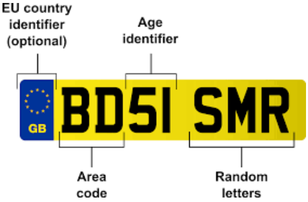
Permutations with repetition

The *r-permutations with repetition* of an alphabet $A = \{a_1, \ldots, a_n\}$ are the ordered sequences of symbols of the form

$$\langle x_1, \ldots, x_r \rangle$$

with $x_1, \ldots, x_r \in A$.

- The same symbol of A can occur many times
- By the rule of product, there are n^r r-permutations with repetition



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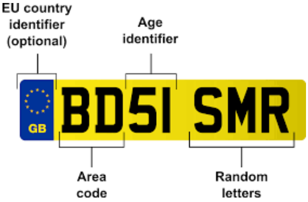

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2. Next, compute how many two-digit year identifiers are possible;

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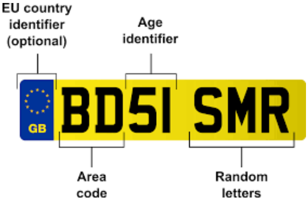

1. If we consider that the year is set, then we compute how many of the random three-letter sequences are possible at the end of the licence plate;
2. Next, compute how many two-digit year identifiers are possible;
3. Next, compute how many of the two-letter area codes, followed by two-digit year identifiers, followed by the random three-letter sequences are possible.

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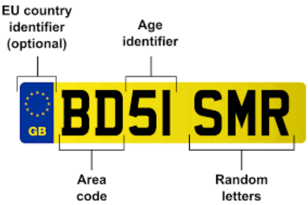

1. If we consider that the year is set, then we compute how many of the random three-letter sequences are possible at the end of the licence plate;
2. Next, compute how many two-digit year identifiers are possible;
3. Next, compute how many of the two-letter area codes, followed by two-digit year identifiers, followed by the random three-letter sequences are possible.
4. The result is ???

The r-permutations with repetition can be ordered lexicographically:

- $\langle x_1, \ldots, x_r \rangle < \langle y_1, \ldots, y_r \rangle$ if there exists $k \in \{1, \ldots, n\}$ such that $x_k < y_k$ and $x_i = y_i$ for all $1 \leq i < k$.

Example ($A = \{a_1, a_2\}$ with $a_1 < a_2$, and $r = 3$)

r-permutation with repetition of A	lexicographic rank
$\langle a_1, a_1, a_1 \rangle$	0
$\langle a_1, a_1, a_2 \rangle$	1
$\langle a_1, a_2, a_1 \rangle$	2
$\langle a_1, a_2, a_2 \rangle$	3
$\langle a_2, a_1, a_1 \rangle$	4
$\langle a_2, a_1, a_2 \rangle$	5
$\langle a_2, a_2, a_1 \rangle$	6
$\langle a_2, a_2, a_2 \rangle$	7

Let $A = \{a_1, a_2, \ldots, a_n\}$ with $a_1 < a_2 < \ldots < a_n$.

- If we define $index(a_i) := i - 1$ for $1 \leq i \leq n$, and replace a_i with $index(a_i)$ in the lexicographic enumeration of the r-permutations, we get

r-permutation with repetition	encoding as number in base n	lexicogaphic rank
$\langle a_1, \ldots, a_1, a_1, a_1 \rangle$	$\langle 0, \ldots, 0, 0, 0 \rangle$	0
$\vdots$	$\vdots$	$\vdots$
$\langle a_1, \ldots, a_1, a_1, a_n \rangle$	$\langle 0, \ldots, 0, 0, n-1 \rangle$	$n-1$
$\langle a_1, \ldots, a_1, a_2, a_1 \rangle$	$\langle 0, \ldots, 0, 1, 0 \rangle$	n
$\vdots$	$\vdots$	$\vdots$
$\langle a_1, \ldots, a_1, a_2, a_n \rangle$	$\langle 0, \ldots, 0, 1, n-1 \rangle$	$2\,n-1$
$\vdots$	$\vdots$	$\vdots$

REMARK: The r-permutation with repetition of the indexes is the representation in base n of its lexicographic rank.

1. How would you approach the vehicle plates problem differently if letters and digits could not occur repeatedly? Explain.
2. How many different licence plates are possible in ROMANIA based on the license plate set up?
3. Define an algorithm which computes the rank of the r-permutation with repetition $\langle x_1, \ldots, x_r \rangle$ of $A = \{1, \ldots, n\}$ with respect to the lexicographic order.
4. Define an algorithm which computes r-permutation with repetition $\langle x_1, \ldots, x_r \rangle$ with rank k of $A = \{1, \ldots, n\}$ with respect to the lexicographic order.
5. Define an algorithm which computes the r-permutation with repetition immediately after the r-permutation with repetition $\langle x_1, \ldots, x_r \rangle$ of A, in lexicographic order.

Alex is a second year bachelor student at Faculty of Mathematics and Informatics from West University of Timisoara. He fell in love with a colleague of his own year and after a few days he decided that it's time to declare her his love. So, he went to the flower shop to buy her a bouquet of 7 roses. The flower shop had white, yellow and red roses. And because he is a computer scientist, he asked himself:

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1. to compute the number of all possible bouquets;
2. to write all the bouquet options.

The Problem of roses

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The Problem of roses

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4. How to generate all the bouquet options?

An r-combination of a set $A = \{a_1, a_2, \ldots, a_n\}$ is a subset with r elements of A.

There is a bijective correspondence between the set of n-bit strings and the set of subsets of A:

$$B \subseteq A \mapsto b_{n-1}b_{n-2}\ldots b_0 \quad \text{where } b_i = \begin{cases} 1 & \text{if } a_{n-i} \in B \\ 0 & \text{otherwise.} \end{cases}$$

n-bit string $b_0 b_1 \ldots b_{n-1} \mapsto$ subset $\{a_{n-i} \mid b_i = 1\}$ of A

Example

$A = \{a_1, a_2, a_3, a_4, a_5\}$ where $a_1 = a, a_2 = b, a_3 = c, a_4 = d, a_5 = e$.

$\emptyset \leftrightarrow 00000$	$\{a, b\} \leftrightarrow 00011$	$\{c, d, e\} \leftrightarrow 11100$
$\{a\} \leftrightarrow 00001$	$\{a, c\} \leftrightarrow 00101$	$\{b, c, d, e\} \leftrightarrow 11110$
$\{b\} \leftrightarrow 00010$	$\{a, d\} \leftrightarrow 01001$	$\{a, b, d, e\} \leftrightarrow 11011$
$\{c\} \leftrightarrow 00100$	$\{a, e\} \leftrightarrow 10001$	$\{a, c, d, e\} \leftrightarrow 11101$
$\{d\} \leftrightarrow 01000$	$\{b, c\} \leftrightarrow 00110$	$\{a, b, c, d\} \leftrightarrow 01111$
$\{e\} \leftrightarrow 10000$	$\ldots$	$\{a, b, c, d, e\} \leftrightarrow 11111$

The n-bit string encoding of a subset

```
BitString(B: subset of A,
          A:  ordered set {a_1, ..., a_n})
int bit_string[0 .. n − 1]
for i:=0 to n − 1 do
    if a_i ∈ B then
      bit_string[n − i] := 1
    else
      bit_string[n − i] := 0
return bit_string
```

The subset of an n-bit string encoding

```
Combination(b[0..n-1]: bit string,
            A: ordered set {a_1,...,a_n})
B:=∅
for i:=0 to n-1 do
    if b[i] = 1 then
        add a_{n-i} to B
return B
```

The ordering of combinations via bit string encodings

There is a bijective correspondence between the n-bit string encodings and the numbers from 0 to $2^n - 1$:

- n-bit-string $b[0\mathinner{..}n-1] \mapsto$ number $\displaystyle\sum_{i=0}^{n-1} b[i] \cdot 2^i \in \{0, 1, \ldots, 2^n - 1\}$
- number $0 \leq r < 2^n \mapsto$ n-bit-string $b[0\mathinner{..}n-1]$ where

$$b[i] := \left\lfloor \frac{c_i}{2^i} \right\rfloor \text{ where } c_i \text{ is the remainder of dividing } r \text{ with } 2^{i+1}.$$

Definition

The canonic rank of a a subset B of an ordered set A with n elements is

$$\text{CanonicRank}(B, A) := \sum_{i=0}^{n-1} b[i] \cdot 2^i$$

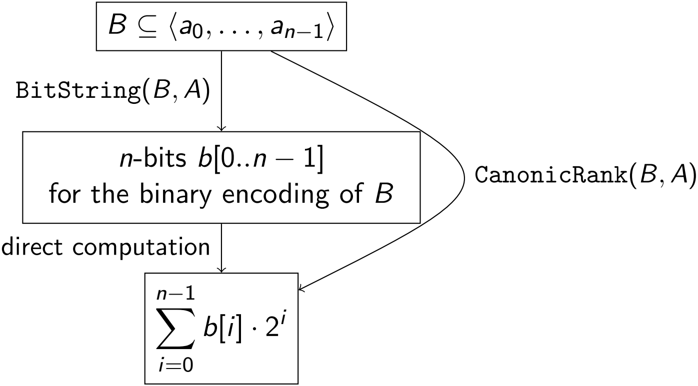
where $b[0\mathinner{..}n-1]$ is the n-bit-string encoding of B as subset of A.

Example ($A = \{a_0, a_1, a_2\}$)

subset	3-bit string encoding $b_2 b_1 b_0$	canonic rank
$\emptyset$	000	0
$\{a_0\}$	001	1
$\{a_1\}$	010	2
$\{a_0, a_1\}$	011	3
$\{a_2\}$	100	4
$\{a_0, a_2\}$	101	5
$\{a_1, a_2\}$	110	6
$\{a_0, a_1, a_2\}$	111	7

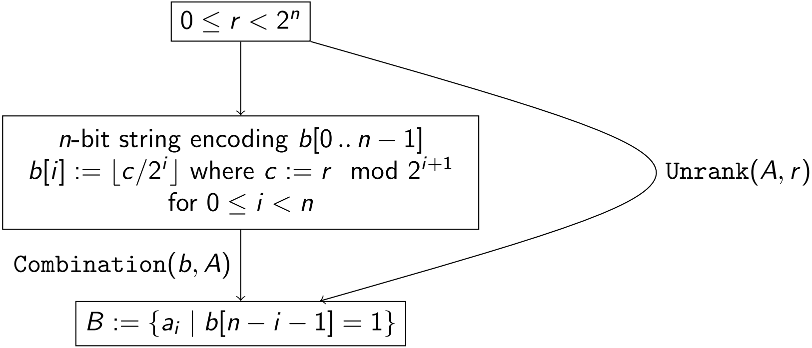
REMARK. This way of enumerating the subsets of a set is called canonic ordering, and the 3-bit string $b_2 b_1 b_0$ is called canonic (or binary) code.

The ordering of combinations via bit string encodings (2)



Given an ordered set $A = \{a_0, a_1, \ldots, a_{n-1}\}$, and $0 \leq r < 2^n$

Find the subset B of A with rank r



- Frank Grey discovered in 1953 a method to enumerate subsets in an order so that adjacent subsets differ by the insertion or deletion of only one element.
- His enumeration scheme is called standard reflected Grey code.

Example

With Grey's method, the subsets of $\{a, b, c\}$ are enumerated in the following order:

$$\{\}, \{c\}, \{b, c\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, c\}, \{a\}$$

The 3-bit-string encodings $b_0 b_1 b_2$ of these subsets are

$$000, 100, 110, 010, 011, 111, 101, 001$$

We want to enumerate the subsets of $A = \{a_1, \ldots, a_n\}$ in minimum change order G_n. (G_n is the list of those subsets)

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We proceed recursively:

1. Compute the list G_{n-1} of subsets of $B = \{a_2, \ldots, a_n\}$ in the minimum change order of Gray.
2. Let G'_{n-1} be the list of subsets obtained by adding a_1 to every element of a reversed copy fo G_{n-1}.
3. G_n is the concatenation of G_{n-1} with G'_{n-1}.

Properties of Grey's codes reflected

Assume that B is a subset of the ordered set A with n elements. If

- m is the rank of B in the order of the Grey's enumeration and $m = \sum_{i=0}^{n-1} b_i \cdot 2^i$
- The codification as a n bit string of B is $c_0 c_1 \ldots c_{n-1}$

then

- $c_i = (b_i + b_{i+1}) \mod 2 \quad \text{for all } 0 \leq i < n,$ where $b_n = 0$.
- On the other hand, one can prove that

$$b_i = (c_i + c_{i+1} + \ldots + c_{n-1}) \mod 2 \quad \text{for all } 0 \leq i < n.$$

Grey's codes

Example ($A = \{a, b, c\}$ with $a < b < c$)

subset B	Grey rank m	$b_0 b_1 b_2$ such that $m = \sum_{i=0}^{2} b_{2-i} 2^i$	bit string of B $c_0 c_1 c_2$	rank of B
$\{\}$	0	000	000	0
$\{c\}$	1	100	100	4
$\{b, c\}$	2	010	110	6
$\{b\}$	3	110	010	2
$\{a, b\}$	4	001	011	3
$\{a, b, c\}$	5	101	111	7
$\{a, c\}$	6	011	101	5
$\{a\}$	7	111	001	1

Notice that $c_i = (b_i + b_{i+1}) \mod 2$ for all $0 \leq i < 3$, where $b_3 = 0$.

Exercises

1. Use the equations in the previous slide to implement the ordering method `RankGrey(B,A)` and the enumeration method `UnrankGrey(A,r)` for enumerating the subsets based on Grey's codes.
2. Define the method `NextGreyRankSubset(A,B)` which computes the subset of A which is the immediately next one after the subset B in the enumeration of subsets based on Grey's codes.

Given an ordered set A with n elements and $0 \leq k \leq n$.

Generate all the k-combinations of A.

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Method 1 (naive and inefficient): generate and test

1. Generate all the 2^n subsets of A
2. Eliminate the generated subsets which do not have k elements.

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Method 1 (naive and inefficient): generate and test

1. Generate all the 2^n subsets of A
2. Eliminate the generated subsets which do not have k elements.

Method 2 (simple recursion): If $A = \{a\} \cup B$ where $a \notin B$ is the smallest element of A then

1. Generate the list L_1 of all $(k-1)$-combinations of B, and let L_2 be the list of all k-combinations of B.
2. Let L_3 be the list obtained by adding a to all the elements of L_1.
3. Return the result of the concatenation of L_2 with L_3.

Assume $A = \{1, 2, \ldots, n\}$ and $X = \{x_1, x_2, \ldots, x_k\} \subseteq A$ such that $x_1 < x_2 < \ldots < x_k$.

Q: Which is the rank of X in the lexicographic enumeration of the k-combinations of A?

The k-combinations which occur before X in lexicographic order are of 2 kinds:

1. The ones which contain an element smaller than x_1.
2. The ones which contain the minimum element x_1, but the rest of the elements is a $(k-1)$-combination smaller than $\{x_2, x_3, \ldots, x_k\}$.

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$\Rightarrow$ the rank of X in the lexicographic enumeration of the k-combinations of A is $N_1 + N_2$ where

- N_1 is the number of k-combinations of the first kind
- N_2 is the number of the k-combinations of the second kind

HYPOTHESIS: $A = \{1, 2, \ldots, n\}$.

How can we compute N_1?

The lexicographic ordering of k-combinations
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- The number of k-combinations of A which contain i as the smallest element is $\binom{n-i}{k-1} \Rightarrow N_1 = \sum_{i=1}^{x_1-1} \binom{n-i}{k-1}$ (the sum rule)

The lexicographic ordering of k-combinations
Preliminary remarks (2)

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  $\Rightarrow N_1 = \sum_{i=1}^{x_1-1} \left(\binom{n-i+1}{k} - \binom{n-i}{k} \right) = \binom{n}{k} - \binom{n-x_1+1}{k}$

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How can we compute N_2?

- N_2 is the rank of $\{x_2, \ldots, x_k\}$ in the lexicographic enumeration of the $(k-1)$-combinations of $\{x_1 + 1, x_1 + 2, \ldots, n-1, n\}$

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- $\Rightarrow N_2$ can be computed recursively.

From the previous remarks results the following recursive implementation for computing the rank:

- $\texttt{RankKSubset}(\{x_1,\ldots,x_k\},\{\ell,\ldots,n\})$ computes the rank in lexicographic order of the k-combination $\{x_1,\ldots,x_k\}$ of the ordered set $\{\ell,\ell+1,\ldots,n-1,n\}$. Assume that $x_1 < x_2 < \ldots < x_k$.

```
RankKSubset({x_1,...,x_k}, {ℓ,ℓ+1,...,n})
  if (n = k or k=0)
    return 0,
    p := x_1 − ℓ + 1
    if (k = 1)
       return p − 1
    else
       return (n choose k) − (n−p+1 choose k) + RankKSubset({x_2,...,x_k},{x_1 + 1,...,n})
```

Hypothesis:

- $A = \{1, 2, \ldots, n\}$ and $X = \{x_1, x_2, \ldots, x_k\}$ with $x_1 < x_2 < \ldots < x_k$ is the subset of A with rank m in the lexicographic enumeration of all k-combinations of A.
  [Keep in mind that $0 \leq m < \binom{n}{k}$.]

Q̂: Which are the values $x_1, x_2, \ldots, x_k$?

1. The total number of k-combinations of A which contain the element $< x_1$ is

$$\sum_{i=1}^{x_1-1} \binom{n-i}{k-1} = \binom{n}{k} - \binom{n-x_1+1}{k} \leq m. \qquad (1)$$

where $\binom{n-i}{k-1}$ is the number of k-combinations in which the smallest element is $i \in \{1, \ldots, x_1 - 1\}$. This number is $\leq m$ because all these k-combinations are lexicographic smaller than X, which has the rank m.

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2. The total number of k-combinations of A which contain an element $\leq x_1$ is

$$\sum_{i=1}^{x_1} \binom{n-i}{k-1} = \binom{n}{k} - \binom{n-x_1}{k} > m. \tag{2}$$

where $\binom{n-i}{k-1}$ is the number of k-combinations in which the smallest element is $i \in \{1, \ldots, x_1\}$. This number is $> m$ because there are $m+1$ integers i between 0 and the rank of X (which is m), and all the k-combinations with such a rank i contain one element $\leq x_1$.

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$\Rightarrow$ one can use (1) and (2) to find x_1: $\binom{n}{k} - \binom{n-x_1+1}{k} \leq m < \binom{n}{k} - \binom{n-x_1}{k}$

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The other elements $x_2, \ldots, x_k$ can be computed recursively.

`UnrankKSubset`$(m, k, \{a_1, \ldots, a_n\})$ produces the k-combination $\{x_1, \ldots, x_k\}$ with rank m of $\{a_1, \ldots, a_n\}$ in lexicographic order. Assume that $x_1 < \ldots < x_k$ and $a_1 < \ldots < a_n$.

`UnrankKSubset`$(m, k, \{a_1, \ldots, a_n\})$
if $(k = 1)$
    return a_{k+1}
else if $(m = 0)$
    return $\{a_1, \ldots, a_m\}$
else
    $u := \binom{n}{k}$
    $i := 1$
    while $\binom{i}{k} < u - m$
        i++
    $x1$:=$n - (i - 1)$
    return $\{a_{n-i+1}\} \cup$ `UnrankKSubset`$(m - u + \binom{n-x1+1}{k}, k - 1, \{a_{n-i+2}, \ldots, a_n\})$

Solution for the Problem of roses
(slide 11)

Now that we have all the necessary informations, we get back to the Problem of roses.

1. How many different flowers bouquets can be obtained for Alex's future girlfriend?
2. Which one is the most beautiful?

Answers ???

References

- S. Pemmaraju, S. Skiena. *Combinatorics and Graph Theory with Mathematica*. Section 2.3: Combinations. Cambridge University Press. 2003.