Lecture 13

Planar graphs.
Graph colourings. Chromatic polynomials

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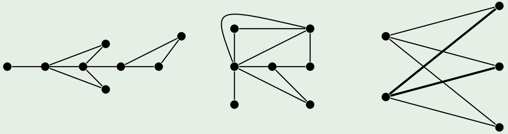
January 2018

Planar graphs
Definition and examples

A graph G is planar if it can be drawn in the plane such that pairs of edges intersect only at vertices, if at all. Such a drawing is called planar representation of G.

Example (Planar graphs)



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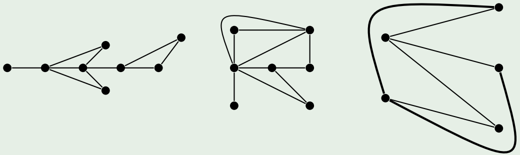
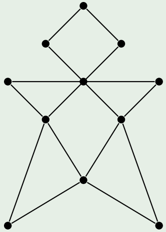


Auxiliary notions

Region of a planar representation of a graph G: maximal section of the plane in which any two points can be joined by a curve that does not intersect any part of G.

Example



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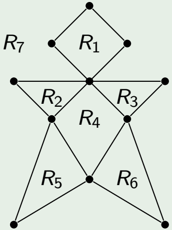

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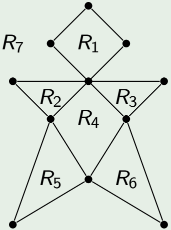

determines 7 regions
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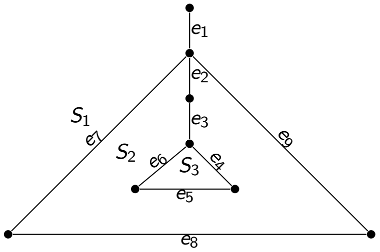
- Every region is delimited by edges.
- An edge is in contact with one or two regions.
- An edge borders a region R if it is in contact with R and with another region.

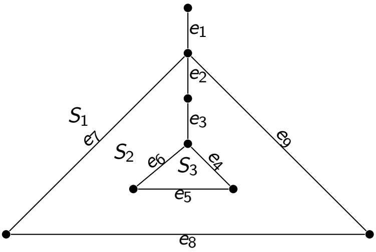
Regions and bound degrees



e_1 is in contact only with S_1
e_2 and e_3 are in contact only with S_2

S_1 is bordered by e_7, e_8, e_9
S_3 is bordered by e_4, e_5, e_6
S_2 is bordered by $e_4, e_5, e_6, e_7, e_8, e_9$

Regions and bound degrees



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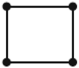
The bound degree $b(S)$ of a region S is the number of edges that border S.

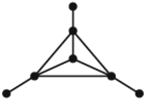
$$b(S_1) = 3, \quad b(S_2) = 6, \quad b(S_3) = 3$$

Properties

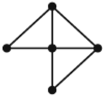
Let G be a connected graph with n nodes, q edges, and a planar representation of G with r regions.



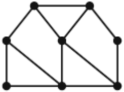
$n = 4$
$q = 4$
$r = 2$



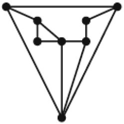
$n = 7$
$q = 9$
$r = 4$



$n = 5$
$q = 7$
$r = 4$



$n = 8$
$q = 12$
$r = 6$



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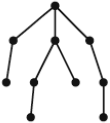

$n = 10$
$q = 9$
$r = 1$

Properties

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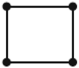

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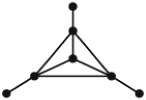

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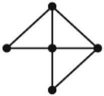

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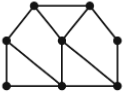

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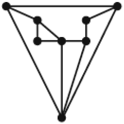

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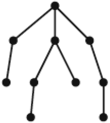

$n = 10$
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$n - q + r = 2$ in all cases.

Properties of connected planar graphs

Theorem (Euler's Formula)

If G is a connected planar graph with n nodes, q edges and r regions then $n - q + r = 2$.

PROOF: Induction on q.

CASE 1: $q = 0 \Rightarrow G = K_1$ and $n = 1$, $q = 0$, $r = 1$, thus $n - q + r = 2$.

CASE 2: G is a tree $\Rightarrow q = n - 1$ and $r = 1$, thus $n - q + r = n - (n - 1) + 1 = 2$.

CASE 3: G is a connected graph with a cycle. Let e be an edge of that cycle, and $G' = G - e$.

G' is connected with n nodes, $q - 1$ edges, and $r - 1$ regions $\Rightarrow$ by Induction Hypothesis: $n - (q - 1) + (r - 1) = 2$.

Thus $n - q + r = 2$ holds in this case too.

Consequences of Euler's Formula

Corollary 1

$K_{3,3}$ is not planar.

PROOF: $K_{3,3}$ has $n = 6$ and $q = 9 \Rightarrow$ if it were planar, it would have $r = q - n + 2 = 5$ regions R_i $(1 \leq i \leq 5)$. Let $C = \sum_{i=1}^{5} b(R_i)$.

- Every edge is in contact with at most 2 regions $\Rightarrow C \leq 2\,q = 18$.
- $K_{3,3}$ is bipartite $\Rightarrow C_3$ is no subgraph of $K_{3,3}$, thus $b(S_i) \geq 4$ for all i, therefore $C \geq 4 \cdot 5 = 20$

$\Rightarrow$ contradiction, thus $K_{3,3}$ can not be planar.

Consequences of Euler's Formula

Corollary 2

If G is a planar graph with $n \geq 3$ nodes and q edges then $q \leq 3\,n - 6$. Moreover, if $q = 3\,n - 6$ then $b(S) = 3$ for every region S of G.

PROOF. Let $R_1, \ldots, R_r$ be the regions of G and $C = \sum_{i=1}^{r} b(R_i)$. We know that $C \leq 2\,q$ and $C \geq 3r$ (because $b(R_i) \geq 3$ for all i). Therefore
$3\,r \leq 2\,q \Rightarrow 3\,(2 + q - n) \leq 2\,q \Rightarrow q \leq 3n - 6$.
If the equality holds, then
$3\,r = 2\,q \Rightarrow C = \sum_{i=1}^{r} b(R_i) = 3\,r \Rightarrow b(R_i) = 3$ for all regions R_i.

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Corollary 3

K_5 is not planar.

PROOF: K_5 has $n = 5$ nodes and $q = 10$ edges $\Rightarrow 3\,n - 6 = 9 < 10 = q$
$\Rightarrow K_5$ can not be planar (Cf. Corollary 2).

Consequences of Euler's Formula

Corollary 4

$\delta(G) \leq 5$ for every planar graph G.

PROOF: Suppose G has n nodes and q edges.

CASE 1: $n \leq 6 \Rightarrow$ every node has degree $\leq 5 \Rightarrow \delta(G) \leq 5$.

CASE 2: $n > 6$. Let $D = \sum_{v \in V} \deg(v)$. Then

$$\begin{aligned} D &= 2\,q && \text{(obvious)} \\ &\leq 2\,(3\,n - 6) && \text{(by Corollary 2)} \\ &= 6\,n - 12. \end{aligned}$$

If $\delta(G) \geq 6$ then $D = \sum_{v \in V} \deg(v) \geq \sum_{v \in V} 6 = 6\,n$, contradiction.

Thus $\delta(G) \leq 5$ holds.

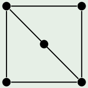
Subdivisions

Let $G = (V, E)$ be an undirected graph, and (x, y) an edge.

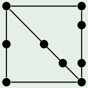
- A subdivision of (x, y) in G is a replacement of the edge (x, y) in G with a path from x to y through some new intermediate points.
- A graph H is a subdivision of a graph G if H can be produced from G through a finite sequence of edge divisions.

Example

G :



H :



Criteria to detect planar graphs

We say that a graph G contains a graph H if H can be produced by removing edges and nodes from G.

Remark

If H is a subgraph of G then G contains H. The converse is false: "G contains H" does not imply "H is a subgraph of G".

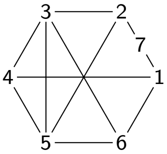
- H is a subgraph of G iff it can be produced from G by node removals.

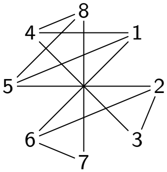
Theorem (Kuratowski's Theorem)

G is planar if and only if it contains no subdivisions of $K_{3,3}$ and of K_5.

Kuratowski's Theorem
Illustrated examples

Apply Kuratowski's Theorem to decide which of the following graphs are planar or not:

1. 

2. 

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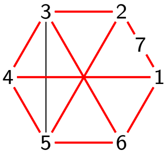
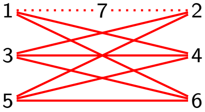


No, because it contains a subdivision of $K_{3,3}$:



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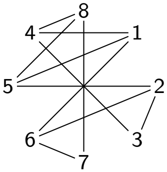

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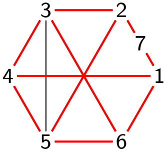

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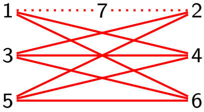

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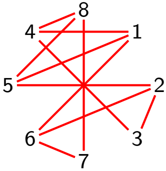

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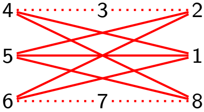

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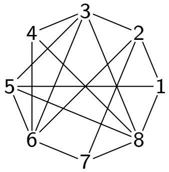
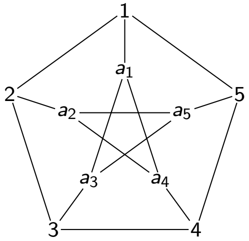
3. 

4. 

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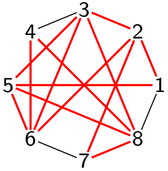

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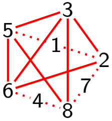

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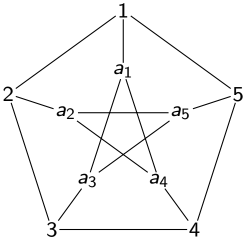

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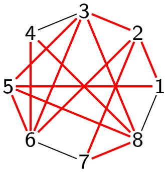

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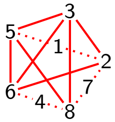

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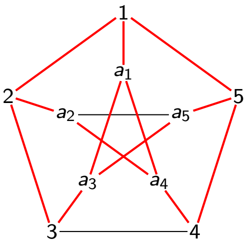

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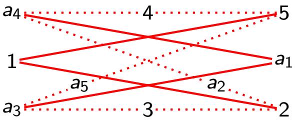


Motivating problem

Alan, Bob, Carl, Dan, Elvis, Ford, Greg and John are senators which comprise 7 committees:

$C_1 = \{$Alan, Bob, Carl$\}$, $C_2 = \{$Carl, Dan, Elvis$\}$,
$C_3 = \{$Dan,Ford$\}$, $C_4 = \{$Adam, Greg$\}$, $C_5 = \{$Elvis, John$\}$,
$C_6 = \{$Elvis,Bob,Greg$\}$, $C_7 = \{$John, Carl, Ford$\}$.

Every committee must fix a meeting time. Since each senator must be present at each of his or her committee meetings, the meeting times need to be scheduled carefully.

Question: What is the minimum number of meeting times?

Answer to the motivating problem

Remarks:

- Two committees C_i and C_j can not meet simultaneously if and only if they have a common member (i.e., $C_i \cap C_j = \emptyset$).

$\Rightarrow$ we can consider the undirected graph G with

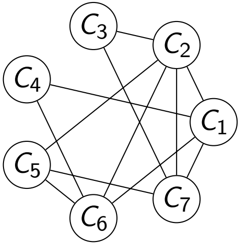
  - nodes = the committees $C_1, C_2, C_3, C_4, C_5, C_6, C_7$
  - edges (C_i, C_j) if C_i and C_j share a member (i.e., $C_i \cap C_j = \emptyset$)

- We color every node C_i with a colour representing its meeting time C_i

  $\Rightarrow$ the problem is reduced to: what is the minimum number of colours that can be assigned to the nodes of G, such that no edge has endpoints with the same colours?

Answer to the motivating problem

G :



Definition (node colouring, chromatic number)

A *k*-colouring of the nodes of a graph $G = (V, E)$ is a map $K : V \to \{1, \ldots, k\}$ such that $K(u) \neq K(v)$ if $(u, v) \in E$.
The chromatic number $\chi(G)$ of a graph G is the minimum value of $k \in \mathbb{N}$ for which there exists a k-colouring of G.

Answer to the motivating problem

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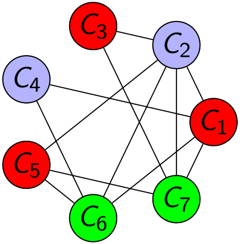

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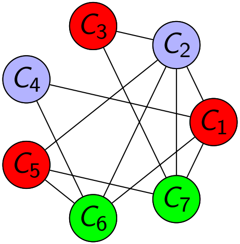

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$\Rightarrow$ the minimum number is 3.
(we need 3 colours)

Definition (node colouring, chromatic number)

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Chromatic polynomials

The computation of $\chi(G)$ is a hard problem (NP-complete).

- Birkhoff ($\approx$ 1900) found a method to compute a polynomial $c_G(z)$ for any graph G, called the chromatic polynomial of G, such that
  - $c_G(k) =$ the number of k-colourings of the nodes of G

$\Rightarrow$ $\chi(G) =$ niminum value of k for which $c_G(k) > 0$.

Chromatic polynomials

The computation of $\chi(G)$ is a hard problem (NP-complete).

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We will present

1. simple formulas of $c_G(z)$ for some special graphs G.
2. two recursive algorithms for the computation of $c_G(z)$ for any graph G.

Chromatic polynomials for special graphs

1. The empty graph E_n: 

every node can be coloured with any of the z available colours:
$\Rightarrow c_{E_n}(z) = z^n$ and $\chi(E_n) = 1$

Chromatic polynomials for special graphs

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2. Tree T_n with n nodes:
   - z alternatives to colour the root node
   - any other node can be coloured with any colour different from the colour of the parent node $\Rightarrow z - 1$ alternatives to colour it

   $$\Rightarrow c_{T_n}(z) = z \cdot (z-1)^{n-1} \text{ and } \chi(T_n) = \begin{cases} 1 & \text{if } n = 1, \\ 2 & \text{if } n > 1. \end{cases}$$

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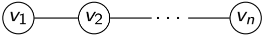
3. Special case: the graph P_n (path with n nodes) is a special tree with n nodes: 

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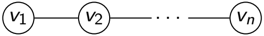
3. Special case: the graph P_n (path with n nodes) is a special tree with n nodes: 

   $\Rightarrow c_{P_n}(z) = z \cdot (z-1)^{n-1}$ and $\chi(P_n) = \begin{cases} 1 & \text{if } n = 1, \\ 2 & \text{if } n > 1. \end{cases}$
4. Complete graph K_n:
   $c_{K_n}(z) = z \cdot (z-1) \cdot \ldots \cdot (z-n+1)$ and $\chi(K_n) = n$.

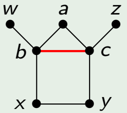
The computation of chromatic polynomials
Special operations on graphs

LEt $G = (V, E)$ be an undirected graph, and $e = (x, y)$ an edge from E

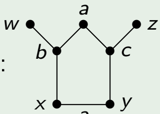
- $G - e$ is the graph produced from G by removing e
- G/e is the graph produced G as follows:
  - Collapse x and y into one node, whose neighbours are the previous neighbours of x and y.

Example

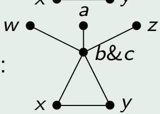
G:



$G - (b, c)$:



$G/(b, c)$:



The computation of chromatic polynomials

Recursive formulas

Note, that, for every $e \in E$: $c_G(z) = c_{G-e}(z) - c_{G/e}(z)$
$\Rightarrow$ two algorithms for the recursive computation of the chromatic polynomial:

1. Reduce G by eliminating edges $e \in E$ one by one:

$$c_G(z) = c_{G-e}(z) - c_{G/e}(z)$$

   until we reach special polynomials E_n or T_n:
   - Base cases: $c_{E_n}(z) = z^n$ and $c_{T_n}(z) = z \cdot (z-1)^{n-1}$
2. Extend G by adding edges that are missing from G:

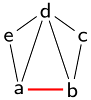
$$c_G(z) = c_{\bar{G}}(z) + c_{\bar{G}/e}(z)$$

   where e is an edge missing from G, and $\bar{G} = G + e$
   - Base case: $c_{K_n}(z) = z \cdot (z-1) \cdot \ldots \cdot (z-n+1)$

The computation of the chromatic polynomial by reduction

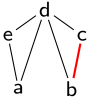
Illustrated example

G :

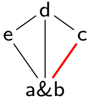


$c_G(z) = c_{G_1}(z) - c_{G_2}(z)$, where

$G_1 = G - (a, b)$:



$G_2 = G/(a, b)$:



The computation of the chromatic polynomial by reduction

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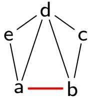

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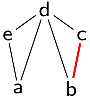

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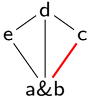
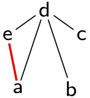
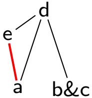


$c_{G_1}(z) = c_{G_{11}}(z) - c_{G_{12}}(z)$ and $c_{G_2}(z) = c_{G_{21}}(z) - c_{G_{22}}(z)$, where

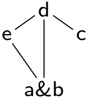
G_{11}:



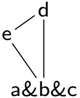
G_{12}:



G_{21}:



G_{22}:



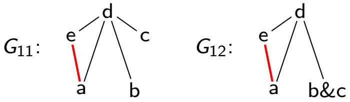
The following graphs are isomorphic: $G_{12} \equiv G_{21}$ and $G_{22} = K_3$, thus:

$$c_G(z) = c_{G_{11}}(z) - 2 \cdot c_{G_{12}}(z) + \underbrace{z(z-1)(z-2)}_{c_{K_3}(z)}$$

The computation of the chromatic polynomial by reduction

Illustrated example (continued)

$$c_G(z) = c_{G_{11}}(z) - 2 \cdot c_{G_{12}}(z) + z(z-1)(z-2)$$



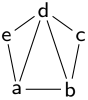
Note that

- $c_{G_{11}}(z) = c_{T_5}(z) - c_{T_4}(z) = z(z-1)^4 - z(z-1)^3$
- $c_{G_{12}}(z) = c_{T_4}(z) - c_{T_3}(z) = z(z-1)^3 - z(z-1^2)$

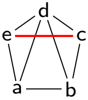
$$\Rightarrow c_G(z) = z(z-1)^4 - z(z-1)^3 - 2(z(z-1)^3 - z(z-1)^2) + z(z-1)(z-2) = z^5 - 7z^4 + 18z^3 - 20z^2 + 8z$$

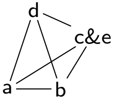
The computation of the chromatic polynomial by extension

Illustrated example

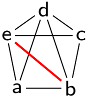
G :


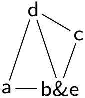
$c_G(z) = c_{G_1}(z) + c_{G_2}(z)$, where

G_1 :


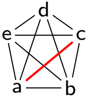
G_2 :


$c_{G_2}(z) = z(z-1)(z-2)(z-3)$ because $G_2 \equiv K_4$, and

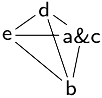
$c_{G_1}(z) = c_{G_{11}}(z) + c_{G_{12}}(z)$ where G_{11} :


G_{12} :


$$c_{G_{11}}(z) = c_{G_{111}}(z) + c_{G_{112}}(z) = c_{K_5}(z) + c_{K_4}(z)$$ where

G_{111} :


$G_{111} \equiv K_5$

G_{112} :


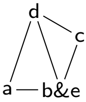
$G_{112} \equiv K_4$

The computation of the chromatic polynomial by extension

Illustrated example (continued)

$$c_G(z) = c_{G_1}(z) + c_{G_2}(z) = (c_{G_{11}}(z) + c_{G_{12}}(z)) + c_{K_4}(z)$$
$$= c_{K_5}(z) + c_{K_4}(z) + c_{G_{12}}(z) + c_{K_4}(z)$$

where G_{12} :



The computation of the chromatic polynomial by extension

Illustrated example (continued)

$$c_G(z) = c_{G_1}(z) + c_{G_2}(z) = (c_{G_{11}}(z) + c_{G_{12}}(z)) + c_{K_4}(z)$$
$$= c_{K_5}(z) + c_{K_4}(z) + c_{G_{12}}(z) + c_{K_4}(z)$$

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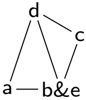
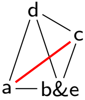


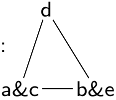
$c_{G_{12}}(z) = c_{G_{121}}(z) + c_{G_{122}}(z) = c_{K_4}(z) + c_{K_3}(z)$ where

where G_{121} :



$G_{121} \equiv K_4$

where G_{122} :



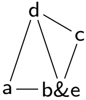
$G_{122} \equiv K_3$

The computation of the chromatic polynomial by extension

Illustrated example (continued)

$$c_G(z) = c_{G_1}(z) + c_{G_2}(z) = (c_{G_{11}}(z) + c_{G_{12}}(z)) + c_{K_4}(z)$$
$$= c_{K_5}(z) + c_{K_4}(z) + c_{G_{12}}(z) + c_{K_4}(z)$$

where G_{12} :



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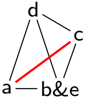

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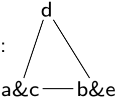


$G_{122} \equiv K_3$

$$\Rightarrow c_G(z) = c_{K_5}(z) + 3c_{K_4}(z) + c_{K_3}(z) = z^5 - 7z^4 + 18z^3 - 20z^2 + 8z$$

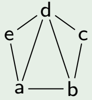
Properties of the chromatic polynomial

If $G = (V, E)$ is an undeirected graph with n nodes and q edges then the chromatic polynomial $c_G(z)$ satisfies the following conditions:

- It has degree n.
- The coefficient of z^n is 1.
- Its coefficients have alternating signs.
- The constant term is 0.
- The coefficient of z^{n-1} is $-q$.

Example

G :



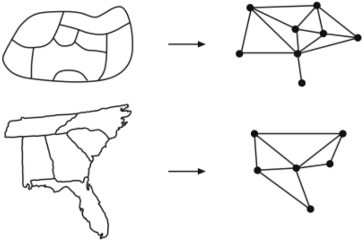
$n = 5, \; q = 7$
$c_G(z) = z^5 - 7\,z^4 + 18\,z^3 - 20\,z^2 + 8\,z$

Remarkable results

Maps and planar graphs

- Every country from a planar map is represented by a node (a point inside it)
- Two nodes get connected if and only if their respective countries share a nontrivial border (mode than just a dot).

⇒ undirected graph G_H corresponding to a map H. For example:



REMARK: ***H* is a planar map if and only if *G_H* is a planar graph.**

Remarkable results

4-colourings of a map

The countries of a planar map H can be coloured with 4 colours, such that no two neighbouring countries have the same colour.

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Remarks

1. This is one of the most famous problems from Graph Theory
   - Extremely long, tedious, and complex proof
   - The problem was proposed in 1858; first proof was given in 1976 (Appel & Haken)
   - The problem is equivalent with the statement that the planar graph G_H is 4-colorable.

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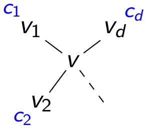
   - Extremely long, tedious, and complex proof
   - The problem was proposed in 1858; first proof was given in 1976 (Appel & Haken)
   - The problem is equivalent with the statement that the planar graph G_H is 4-colorable.
2. This theorem is equivalent with the statement:

$$\chi(G) \leq 4 \text{ for every planar graph } G.$$

5-colourings of planar maps

The countries of a planar map H can be coloured with 5 colours, such that no two neighbouring countries have the same colour. or, equivqlently: $\chi(G) \leq 5$ for every planar graph G.

PROOF: Induction on $n =$ the number of nodes of G.
The statement is obvious for $n \geq 5$, thus we assume $n \geq 6$.
$\delta(G) \leq 5$ by Corollary 4, thus G has a node v with $\deg(v) \leq 5$.
Let G' be the graph produced by removing v from $G \Rightarrow G'$ has $n-1$ nodes, thus $\chi(G') \leq 5$ by Inductive Hypothesis. Therefore, we can assume G' has a 5-colouring with colours 1,2,3,4,5.
CASE 1: $\deg(G) = d \leq 4$. Let $v_1, \ldots, v_d$ be the neighbours of v, with colours $c_1, \ldots, c_d$.

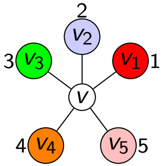


for v we can choose any colour
$c \in \{1,2,3,4,5\} - \{c_1, \ldots, c_d\}$
$\Rightarrow G$ is 5-colourable.

5-colourings of planar maps
Proof (continued)

CASE 2: $\deg(v) = 5$, thus v has 5 neighbours v_1, v_2, v_3, v_4, v_5, which we assume to be coloured with c_1, c_2, c_3, c_4, c_5, respectively.

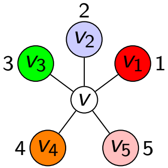
1. If $\{c_1, c_2, c_3, c_4, c_5\} \neq \{1, 2, 3, 4, 5\}$, we can colour v with any colour $c \in \{1, 2, 3, 4, 5\} - \{c_1, c_2, c_3, c_4, c_5\} \Rightarrow G$ is 5-colourable.
2. If $\{c_1, c_2, c_3, c_4, c_5\} = \{1, 2, 3, 4, 5\}$, we can assume $c_1 = 1, c_2 = 2, c_3, c_4 = 4, c_5 = 5$.



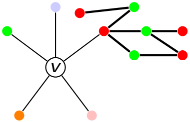
Main idea: We will rearrange the colours of G' in order to make possible a colouring of v.

5-colourings of planar maps

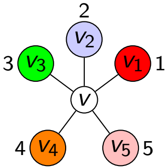
Proof (continued)



We consider all nodes of G' which are coloured with 1 (red) and 3 (green).
CASE 2.1. G' has no path from v_1 to v_3 coloured only with 1 and 3.
Let H be the subgraph of G' made of all paths starting from v_1 which are coloured only with 1 (red) and 3 (green).



5-colourings of planar maps
Proof (continued)



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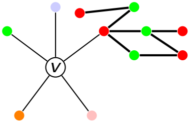
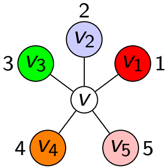
CASE 2.1. G' has no path from v_1 to v_3 coloured only with 1 and 3.
Let H be the subgraph of G' made of all paths starting from v_1 which are coloured only with 1 (red) and 3 (green).

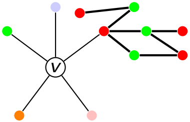


- $V[v_3] \cap V(H) = \emptyset$, that is, neither v_3 nor any of its neighbours is a node of H.

5-colourings of planar maps
Proof (continued)



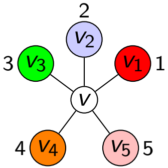
We consider all nodes of G' which are coloured with 1 (red) and 3 (green).
CASE 2.1. G' has no path from v_1 to v_3 coloured only with 1 and 3.
Let H be the subgraph of G' made of all paths starting from v_1 which are coloured only with 1 (red) and 3 (green).



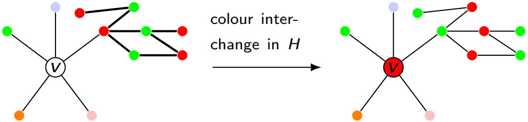
- $V[v_3] \cap V(H) = \emptyset$, that is, neither v_3 nor any of its neighbours is a node of H.
- We can interchange colours 1 and 3 in H, and afterwards assign colour 1 (red) to $v \Rightarrow G$ is 5-colourable.

5-colourings of planar maps

Proof (continued)

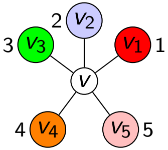


We consider all nodes of G' which are coloured with 1 (red) and 3 (green).
CASE 2.1. G' has no path from v_1 to v_3 coloured only with 1 and 3.
Let H be the subgraph of G' made of all paths starting from v_1 which are coloured only with 1 (red) and 3 (green).

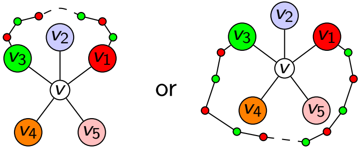


- $V[v_3] \cap V(H) = \emptyset$, that is, neither v_3 nor any of its neighbours is a node of H.
- We can interchange colours 1 and 3 in H, and afterwards assign colour 1 (red) to v $\Rightarrow$ G is 5-colourable.

5-colourings of planar maps
Proof (continued)



CASE 2.2. G' has a path from v_1 to v_3 coloured only with colours 1 and 3 $\Rightarrow$ we are in one of the following two situations:



In both cases, there can be no path from v_2 to v_4 coloured only with 2 and 4 $\Rightarrow$ case 2.1 is applicable to nodes v_2 and v_4 $\Rightarrow$ G is 5-colourable in this case too.