

Lecture 12

Eulerian trails and circuits.
Hamiltonian paths and cycles

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Trails, circuits, paths and cycles

Remember that ...

If $G = (V, E)$ is a simple graph, then

- A **walk** or **path** in G is a sequence of (not necessarily distinct) nodes v_1, v_2, \dots, v_k such that $(v_i, v_{i+1}) \in E$ for $i = 1, 2, \dots, k - 1$. Such a walk is sometimes called a $v_1 - v_k$ walk.
 - v_1 and v_k are the end vertices of the walk.
 - If the vertices in a walk are distinct, then the walk is called a **simple path**.
 - If the edges in a walk are distinct, then the walk is called a **trail**.
- A **cycle** is a simple path v_1, \dots, v_k (where $k \geq 3$) together with the edge (v_k, v_1) .
- A **circuit** or **closed trail** is a trail that begins and ends at the same node.
- The **length** of a **walk** (or simple path, trail, cycle, circuit) is its number of edges, counting repetitions.

Eulerian trails and circuits

- ▶ An **Eulerian trail** in a simple graph $G = (V, E)$ is a trail which includes every edge of G .
- ▶ An **Eulerian circuit** in a simple graph $G = (V, E)$ is a circuit which includes every edge of G .
- ▶ An **Eulerian graph** is a simple graph which contains an Eulerian circuit.

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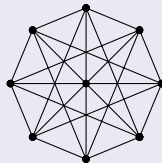
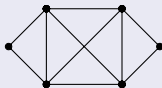
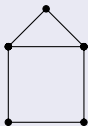
Note that

- ▶ Cycles C_n are Eulerian graphs.
- ▶ Paths P_n have no circuits at all $\Rightarrow P_n$ are not Eulerian graphs.

Eulerian trails and circuits

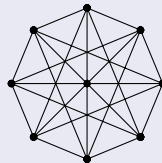
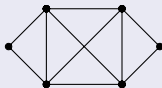
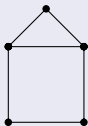
Quiz

Which of the following graphs is Eulerian?



Quiz

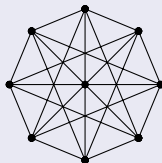
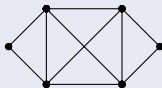
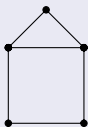
Which of the following graphs is Eulerian?



Q: How can we recognize Eulerian graphs?

Quiz

Which of the following graphs is Eulerian?



Q: How can we recognize Eulerian graphs?

A: Two well-known characterizations:

- 1 based on node degrees
- 2 based on the existence of a special collection of cycles.

Characterization Theorem

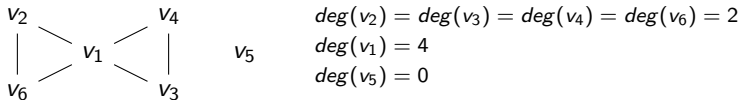
For a connected graph G , the following statements are equivalent:

- 1 G is Eulerian.
- 2 Every vertex of G has even degree.
- 3 The edges of G can be partitioned into (edge-disjoint) cycles.

PROOF OF 1 \Rightarrow 2. Assume

$\triangleright G$ is Eulerian \Leftrightarrow there exists a circuit that includes every edge of G

For example, $v_1, v_3, v_4, v_1, v_2, v_6, v_1$ is a circuit in the graph below:



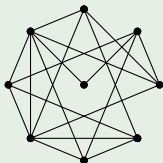
Every time a circuit enters a node v on an edge, it must leave on a different edge. Since the circuit never repeats an edge, the number of edges incident with v is even $\Rightarrow deg(v)$ is even.

Eulerian circuits

Proof of Characterization Theorem (continued)

PROOF OF 2 \Rightarrow 3. Suppose every node of G has even degree. We use induction on the number of cycles in G . G is connected and without nodes of degree 1 $\Rightarrow G$ is not a tree $\Rightarrow G$ has at least one cycle C_{n_1} . Let G' be the graph obtained by removing C_{n_1} from $G \Rightarrow$ all edges of G' have even degree and we can proceed recursively to prove that G' can be partitioned into (edge-disjoint) cycles C_{n_2}, \dots, C_{n_k} . Then $C_{n_1}, C_{n_2}, \dots, C_{n_k}$ is a partition of G into (edge-disjoint) cycles.

Example

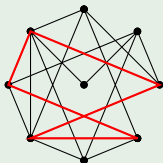


Eulerian circuits

Proof of Characterization Theorem (continued)

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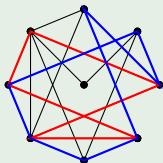


Eulerian circuits

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Example

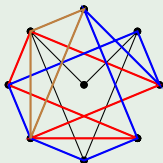


Eulerian circuits

Proof of Characterization Theorem (continued)

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Example

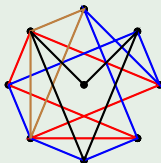


Eulerian circuits

Proof of Characterization Theorem (continued)

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Example

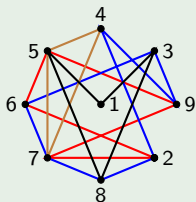


Eulerian circuits

Proof of Characterization Theorem (continued)

PROOF OF 3 \Rightarrow 1. Suppose that the edges of G can be partitioned into k edge-disjoint cycles C_{n_1}, \dots, C_{n_k} . Because G is connected, every such cycle is an Eulerian circuit which must share a node with another cycle \Rightarrow these circuits can be patched until we obtain one Eulerian circuit which is the whole graph G . \square

Example



Cycles:

3,6,7,8,2,4,9,3

3,8,5,1,3

5,6,2,7,9,5

4,5,7,4

- First 2 cycles have common node 3 \Rightarrow circuit
 $S_1 = 3, 8, 5, 1, 3, 6, 7, 8, 2, 4, 9, 3$
- S_1 shares node 6 with 3rd cycle \Rightarrow circuit
 $S_2 = 3, 8, 5, 1, 3, 6, 2, 7, 9, 5, 6, 7, 8, 2, 4, 9, 3$
- Circuit shares node 4 with 4th cycle \Rightarrow Eulerian circuit
 $S_3 = 3, 8, 5, 1, 3, 6, 2, 7, 9, 5, 6, 7, 8, 2, 4, 5, 7, 4, 9, 3$

Finding Eulerian circuits

Hierholzer's Algorithm

The patching algorithm illustrated before is called Hierholzer's Algorithm. It solves the following problem:

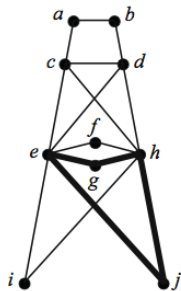
Given: an Eulerian graph G

Find an Eulerian circuit of G .

- 1 Identify a circuit in G and call it R_1 . Mark the edges of R_1 . Let $i = 1$.
- 2 If R_i contains all edges of G , then stop (since R_i is an Eulerian circuit).
- 3 If R_i does not contain all edges of G , then let v_i be a node on R_i that is incident with an unmarked edge, e_i .
- 4 Build a circuit, Q_i , starting at node v_i and using edge e_i . Mark the edges of Q_i .
- 5 Create a new circuit, R_{i+1} , by patching the circuit Q_i into R_i at v_i .
- 6 Increment i by 1, and go to step (2).

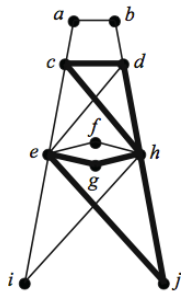
Hierholzer's Algorithm

Illustrated example



$R_1: e, g, h, j, e$

$Q_1: h, d, c, h$



$R_2: e, g, h, d, c, h, j, e$

$Q_2: d, b, a, c, e, d$

Finding Eulerian trails

Question: How can we recognize graphs which contain an Eulerian trail?

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Answer: Note that:

- If the graph is Eulerian, then it contains an Eulerian trail too, because every Eulerian circuit is also a trail.
- **There are non-Eulerian graphs with Eulerian trails.**

Finding Eulerian trails

Question: How can we recognize graphs which contain an Eulerian trail?

Answer: Note that:

- If the graph is Eulerian, then it contains an Eulerian trail too, because every Eulerian circuit is also a trail.
- **There are non-Eulerian graphs with Eulerian trails.**

Corollary

A connected graph G contains an Eulerian trail if and only if there are at most two vertices of odd degree.

Fleury's algorithm

Given a graph G with an Eulerian circuit or path

Find a corresponding circuit or trail.

Initially, all edges are **unmarked**.

- 1 Choose a node v and call it the **lead node**.
- 2 If all edges of G have been marked, then stop. Otherwise continue with next step.
- 3 Among all edges incident with the lead node, choose, if possible, one that is not a bridge of the subgraph formed by the unmarked edges. If this is not possible, choose any edge incident with the lead node. Mark this edge and let its other end node be the new lead node.
- 4 Go to step (2).

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REMARKS:

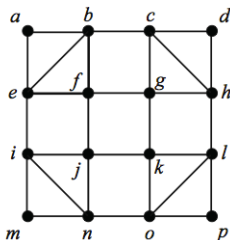
Step 2 is performed $|E|$ times, where $|E|$ is the nr. of edges of G .
In general, detecting if $e \in E$ is a bridge has complexity $O(|E|^2)$

\Rightarrow Fleury's algorithm has complexity $O(|E|^3)$.

Finding Eulerian circuits

Exercises

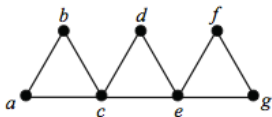
- For each of the following, draw an Eulerian graph that satisfies the conditions, or prove that no such graph exists.
 - An even number of vertices, an even number of edges.
 - An even number of vertices, an odd number of edges.
 - An odd number of vertices, an even number of edges.
 - An odd number of vertices, an odd number of edges.
- Use Hierholzer's algorithm to find an Eulerian circuit in the following graph. Use $R_1 = a, b, c, g, f, j, i, e, a$ as your initial circuit.



Finding Eulerian circuits

Exercises

- 1 Use Fleury's algorithm to find an Eulerian circuit for the graph depicted below. Let a be your initial node.



- 2 Prove that if every edge of a graph G lies on an odd number of cycles, then G is Eulerian.
- 3 Let $G = K_{m,n}$.
 - 1 Find conditions on m and n that characterize when G will have an Eulerian trail.
 - 2 Find conditions that characterize when G will be Eulerian.

Hamiltonian paths and cycles

- A **Hamiltonian path** P of a simple graph G is a simple path that contains all nodes of G .
- A **traceable graph** is a simple graph containing a Hamiltonian path.
- A **Hamiltonian cycle** of a graph is a cycle that contains all nodes of the graph.
- A **Hamiltonian graph** is a graph containing a Hamiltonian cycle.

REMARKS

- 1 All Hamiltonian graphs are traceable.
- 2 There are traceable graphs which are not Hamiltonian.

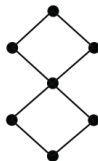
Hamiltonian and traceable graphs

Quiz

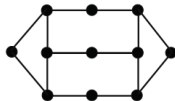
Look at the following graphs and try to determine which ones are traceable, Hamiltonian, or neither.



G_1



G_2



G_3

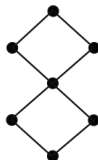
Hamiltonian and traceable graphs

Quiz

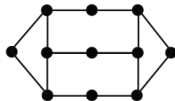
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G_2



G_3

REMARKS. Hamiltonian graphs can have all even degrees (C_{10}), all odd degrees (K_{10}), or a mixture (G_1 in the previous figure).

How to recognize Hamiltonian graphs?

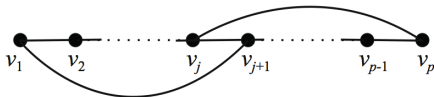
Dirac's Theorem

Dirac's Theorem

Let G be a graph of order $n \geq 3$. If $\delta(G) \geq n/2$, then G is Hamiltonian.

PROOF. Let G satisfy the given conditions. Assume G is not Hamiltonian, and let $P = v_1, \dots, v_p$ be a simple path in G with maximal length. Since P is maximal, all neighbors of v_1 and of v_p are on P . Also, since $\delta(G) \geq n/2$, each of v_1 and v_p has at least $n/2$ neighbors on P .

We claim that $\exists j \in \{1, \dots, p-1\}$ such that $v_j \in N(v_p)$ and $v_{j+1} \in N(v_1)$. If this was not the case, then for every neighbor v_i of v_p on P (and there are at least $n/2$ of them), v_{i+1} is **not** a neighbor of v_1 . This means that $\deg(v_1) \leq p-1 - \frac{n}{2} < n - \frac{n}{2} = \frac{n}{2}$ contradicting the fact that $\delta(G) \geq n/2$. Thus, such a j exists, as shown in the following figure:



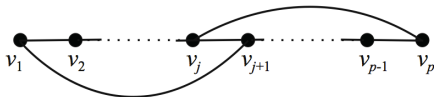
How to recognize Hamiltonian graphs?

Dirac's Theorem (continued)

Dirac's Theorem

Let G be a graph of order $n \geq 3$. If $\delta(G) \geq n/2$, then G is Hamiltonian.

PROOF. (CONTINUED)



Let C be the cycle $v_1, v_2, \dots, v_j, v_p, v_{p-1}, \dots, v_{j+1}, v_1$. Since we assume G is not Hamiltonian, there is a node of G not on P . $\delta(G) \geq n/2$ implies G is connected $\Rightarrow G$ has a node w not on G that is adjacent to some node v_i on P . But then the path starting with w, v_i and then continuing around the cycle C is longer than the maximal path P , contradiction.

We conclude that G is Hamiltonian. □

How to recognize Hamiltonian graphs?

Other criteria and auxiliary notions

Theorem (A generalization Dirac's Theorem)

Let G be a graph of order $n \geq 3$. If $\deg(x) + \deg(y) \geq n$ for all pairs of nonadjacent nodes x, y , then G is Hamiltonian.

How to recognize Hamiltonian graphs?

Other criteria and auxiliary notions

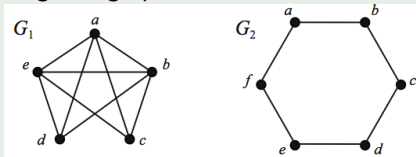
Theorem (A generalization Dirac's Theorem)

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A set of nodes in a graph G is **independent** if they are pairwise nonadjacent. The **independence number** of a graph G , denoted by $\alpha(G)$, is the largest size of an independent set of nodes from G .

Example

Consider the following two graphs



The only independent set of size 2 in G_1 is $\{c, d\}$, so $\alpha(G_1) = 2$. There are two independent sets of size 3 in G_2 : $\{a, c, e\}$ and $\{b, d, f\}$, and none of size 4, so $\alpha(G_2) = 3$.

How to recognize Hamiltonian graphs?

Other criteria and auxiliary notions

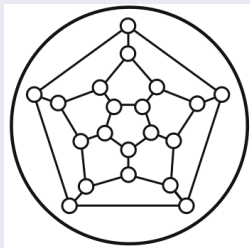
We recall that the vertex connectivity $\kappa(G)$ of a graph G is the minimum size of a node cut set of G .

Theorem (Chvátal and Erdős, 1972)

Let G be a connected graph of order $n \geq 3$ with vertex connectivity $\kappa(G)$ and independence number $\alpha(G)$. If $\kappa(G) \geq \alpha(G)$, then G is Hamiltonian.

Exercise (The Icosian game of Hamilton)

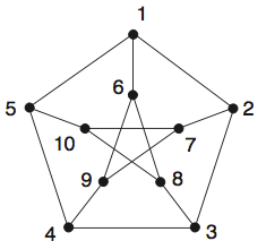
Show that the graph depicted in the circle below is Hamiltonian.



Hamiltonian and traceable graphs

Exercises

- 1 Prove that if G is Hamiltonian, then G is 2-connected.
- 2 Give the connectivity and independence number of the Petersen graph depicted below.

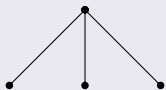


Hamiltonian graphs

Two definitions and 3 special graphs

- Given two graphs G and H , we say that G is **H -free** if G does not contain a copy of H as an induced graph.
- If S is a collection of graphs, we say that G is **S -free** if G does not contain any of the graphs of S as induced subgraph.

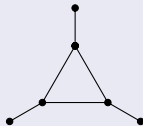
Three special graphs



$K_{1,3}$



Z_1



N

Hamiltonian graphs

Other results

Theorem (Goodman and Hedetniemi, 1974)

If G is a 2-connected, $\{K_{1,3}, Z_1\}$ -free graph, then G is Hamiltonian.

PROOF. Let G be 2-connected and $\{K_{1,3}, Z_1\}$ -free, and let C be a longest cycle in G . Since G is 2-connected, the cycle C exists. We show that C must be Hamiltonian.

If C is not Hamiltonian, there must be a node v not on C that is adjacent to a node w in C . Let a and b be the immediate predecessor and successor of w on C .

- A longer cycle would exist if $\{a, b\} \cap N(v) \neq \emptyset$, thus $\{a, b\} \cap N(v) = \emptyset$.
- If a is not adjacent to b then the subgraph induced by $\{w, v, a, b\}$ is $K_{1,3}$, contradiction with the assumption that G is $K_{1,3}$ -free $\Rightarrow ab$ is an edge in G .
But in this case the subgraph induced by $\{w, v, a, b\}$ is Z_1 , a contradiction with the assumption that G is Z_1 -free.

$\Rightarrow C$ is a Hamiltonian cycle. □

Theorem (Duffus, Gould, and Jacobson, 1981)

Let G be a $\{K_{1,3}, N\}$ -free graph.

- 1 If G is connected, then G is traceable.
- 2 If G is 2-connected, then G is Hamiltonian.

Theorem (Duffus, Gould, and Jacobson, 1981)

Let G be a $\{K_{1,3}, N\}$ -free graph.

- 1 If G is connected, then G is traceable.
- 2 If G is 2-connected, then G is Hamiltonian.

REMARK.

- The graph $K_{1,3}$ is forbidden to appear as a subgraph by both last two theorems. The graph $K_{1,3}$ is usually called the “claw”, and appears as forbidden subgraph in many theorems from graph theory.

- J. M. Harris, J. L. Hirst, M. J. Mossinghoff. *Combinatorics and Graph Theory. Second Edition*. Springer 2008.
Section 1.4. Trails, Circuits, Paths, and Cycles.