

Evolution Strategies

- Particularities
- General structure
- Recombination
- Mutation
- Selection
- Adaptive and self-adaptive variants

Particularities

Evolution strategies: evolutionary techniques used in solving continuous optimization problems

History: the first strategy has been developed in 1964 by Bienert, Rechenberg si Schwefel (students at the Technical University of Berlin) in order to design a flexible pipe

Main ideas [Beyer & Schwefel – ES: A Comprehensive Introduction, 2002]:

- Use one candidate (containing several variables) which is iteratively evolved
- Change all variables at a time, mostly slightly and at random.
- If the new set of variables does not diminish the goodness of the device, keep it, otherwise return to the old status.

Particularities

Data encoding: real (the individuals are vectors of float values belonging to the definition domain of the objective function)

Main operator: mutation (based on parameterized random perturbation)

Secondary operator: recombination

Particularity: self adaptation of the mutation control parameters

General structure

Problem (minimization):

Find x^* in $D \subseteq \mathbb{R}^n$ such that

$f(x^*) < f(x)$ for all x in D

The **population** consists of elements from D (vectors with real components)

Rmk. A configuration is better if the value of f is smaller.

Structure of the algorithm

Population initialization

Population evaluation

REPEAT

construct offspring by recombination

change the offspring by mutation

offspring evaluation

survivors selection

UNTIL <stopping condition>

Resource related
criteria
(e.g.: generations
number, nfe)

Criteria related to the
convergence
(e.g.: value of f)

Recombination

Aim: construct an offspring starting from a set of parents

$$y = \sum_{i=1}^{\rho} c_i x^i, \quad 0 < c_i < 1, \quad \sum_{i=1}^{\rho} c_i = 1$$

Intermediate (convex): the offspring is a linear (convex) combination of the parents

$$y_j = \begin{cases} x_j^1 & \text{with probability } p_1 \\ x_j^2 & \text{with probability } p_2 \\ \vdots & \\ x_j^{\rho} & \text{with probability } p_{\rho} \end{cases},$$

Discrete: the offspring consists of components randomly taken from the parents

$$0 < p_i < 1, \quad \sum_{i=1}^{\rho} p_i = 1$$

Recombination

Geometrical recombination:

$$y_j = (x_j^1)^{c_1} (x_j^2)^{c_2} \dots (x_j^\rho)^{c_\rho}, \quad 0 < c_i < 1, \quad \sum_{i=1}^{\rho} c_i = 1$$

Remark: introduced by Z. Michalewicz for solving constrained optimization problems with constraints involving the product of components (e.g. $x_1 x_2 \dots x_n > c$)

Heuristic recombination:

$y = x^i + u(x^i - x^k)$ with x^i an element at least as good as x^k

u – random value from (0,1)

Recombination

Simulated Binary Crossover (SBX)

- It is a recombination variant which simulates the behavior of one cut point crossover used in the case of binary encoding
- It produces two children c_1 and c_2 starting from two parents p_1 and p_2

Rmk: β is a random value generated according to the distribution given by:

$$c_1 = \bar{p} - \frac{\beta}{2}(p_2 - p_1)$$

$$c_2 = \bar{p} + \frac{\beta}{2}(p_2 - p_1)$$

$$\bar{p} = (p_1 + p_2) / 2$$

$$prob(\beta) = \begin{cases} 0.5(n+1)\beta^n & \beta \leq 1 \\ 0.5(n+1)\frac{1}{\beta^{n+2}} & \beta > 1 \end{cases}$$

Rmk: n can be any natural value; high values of n lead to children which are close to the parents

Mutation

Basic idea: perturb each element in the population by adding a random vector

$$x' = x + z$$

$$z = (z_1, \dots, z_n)$$

random vector with mean 0 and
covariance matrix $C = (c_{ij})_{i,j=1,n}$

Particularity: this mutation favors the small changes of the current element, unlike the mutation typical to genetic algorithms which does not differentiate small perturbations from large perturbations

Mutation

Variants:

- The components of the random vector are independent random variables having the same distribution ($E(z_i z_j) = E(z_i)E(z_j) = 0$).

Examples:

- a) each component is a random value uniformly distributed in $[-s, s]$
- b) each component has the normal (Gaussian) distribution $N(0, s)$

Rmk. The covariance matrix is a diagonal matrix $C = \text{diag}(s^2, s^2, \dots, s^2)$ with s the only control parameter of the mutation

Mutation

Variants:

- The components of the random vector are independent random variables having different distributions ($E(z_i z_j) = E(z_i)E(z_j) = 0$)

Examples:

- a) the component z_i of the perturbation vector has the uniform distribution on $[-s_i, s_i]$
- b) each component of the perturbation vector has the distribution $N(0, s_i)$

Rmk. The covariance matrix is a diagonal matrix:
 $C = \text{diag}(s_1^2, s_2^2, \dots, s_n^2)$ and the control parameters of mutation are s_1, s_2, \dots, s_n

Mutation

Variants:

- The components are dependent random variables

Example:

a) the vector z has the distribution $N(0,C)$

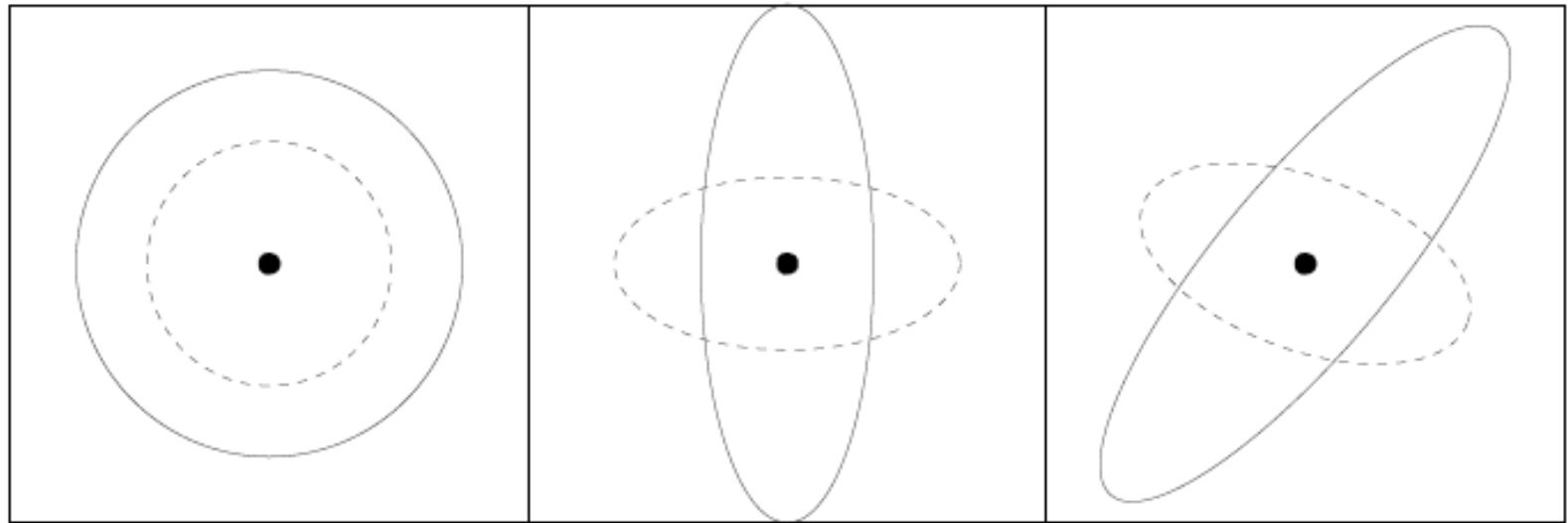
Rmk. There are $n(n+1)/2$ control parameters of the mutation:

s_1, s_2, \dots, s_n - mutation steps

a_1, a_2, \dots, a_k - rotation angles ($k=n(n-1)/2$)

$$c_{ij} = \frac{1}{2} \cdot (s_i^2 - s_j^2) \cdot \tan(2 a_{ij})$$

Mutation



$$\mathcal{N}(m, \sigma^2 \mathbf{I}) \sim m + \sigma \mathcal{N}(\mathbf{0}, \mathbf{I}) \quad \mathcal{N}(m, \mathbf{D}^2) \sim m + \mathbf{D} \mathcal{N}(\mathbf{0}, \mathbf{I}) \quad \mathcal{N}(m, \mathbf{C}) \sim m + \mathbf{C}^{\frac{1}{2}} \mathcal{N}(\mathbf{0}, \mathbf{I})$$

Variants involving various numbers of parameters

[Hansen, PPSN 2006]

Mutation

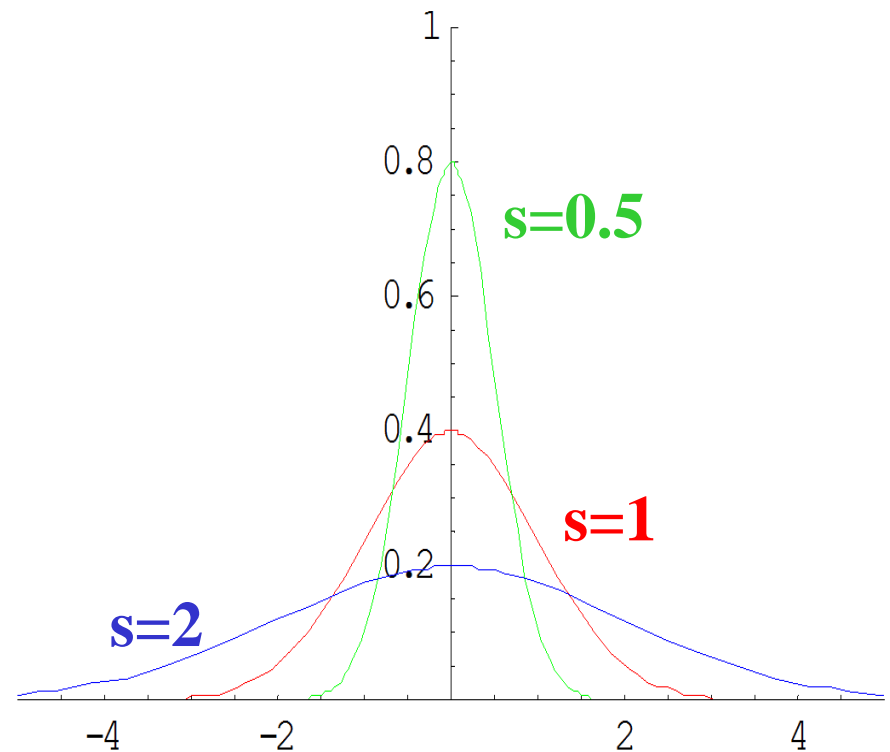
Problem: choice of the control parameters

Example: perturbation of type $N(0,s)$

- s large \rightarrow large perturbation
- s small \rightarrow small perturbation

Solutions:

- Adaptive heuristic methods (example: rule 1/5)
- Self-adaptation (change of parameters by recombination and mutation)



Mutation

1/5 rule.

This is an heuristic rules developed for ES having independent perturbations characterized by a single parameter, s .

Idea: s is adjusted by using the success ratio of the mutation

The success ratio:

$p_s =$ number of mutations leading to better configurations /
total number of mutations

Rmk. 1. The success ratio is estimated by using the results of at least n mutations (n is the problem size)

2. This rule has been initially proposed for populations containing just one element

Mutation

1/5 Rule.

$$s' = \begin{cases} s/c & \text{if } p_s > 1/5 \\ cs & \text{if } p_s < 1/5 \\ s & \text{if } p_s = 1/5 \end{cases}$$

Some theoretical studies conducted for some particular objective functions (e.g. sphere function) led to the remark that c should satisfy $0.8 \leq c < 1$ (e.g.: $c=0.817$)

Remarks:

- This rule was proposed for ESs involving just one candidate; it cannot be directly extended in the case of populations of candidates

Mutation

Self-adaptation

Idea:

- Extend the elements of the population with components corresponding to the control parameters
- Apply specific recombination and mutation operators also to control parameters
- Thus the values of control parameters leading to competitive individuals will have higher chance to survive

Extended population elements

$$\bar{x} = (x_1, \dots, x_n, s)$$
$$\bar{x} = (x_1, \dots, x_n, s_1, \dots, s_n)$$
$$\bar{x} = (x_1, \dots, x_n, s_1, \dots, s_n, a_1, \dots, a_{n(n-1)/2})$$

Mutation

Steps:

- Change the components corresponding to the control parameters
- Change the variables corresponding to the decision variables

Example: the case of independent perturbations

$$\bar{x} = (x_1, \dots, x_n, s_1, \dots, s_n)$$

Variables with lognormal distribution

$$s_i' = s_i \exp(r) \exp(r_i),$$

- ensure that $s_i > 0$

- it is symmetric around 1

$$r \in N(0, 1/\sqrt{2n}), r_i \in N(0, 1/\sqrt{2\sqrt{n}})$$

$$x_i' = x_i + s_i' z \quad \text{with } z \in N(0, 1)$$

Remark:

- The recommended recombination for the control parameters is the intermediate recombination

Mutation

Variant proposed by Michalewicz (1996):

$$x_i'(t) = \begin{cases} x_i(t) + \Delta(t, b_i - x_i(t)) & \text{if } u < 0.5 \\ x_i(t) - \Delta(t, x_i(t) - a_i) & \text{if } u \geq 0.5 \end{cases}$$

$$\Delta(t, y) = y \cdot u \cdot (1 - t/T)^p, \quad p > 0$$

- a_i and b_i are the bounds of the interval corresponding to component x_i
- u is a random value in $(0,1)$
- t is the iteration counter
- T is the maximal number of iterations

Mutation

CMA – ES (Covariance Matrix Adaptation –ES) [Hansen, 1996]

Initialize $\mathbf{m} \in \mathbb{R}^n$, $\sigma \in \mathbb{R}_+$, $\mathbf{C} = \mathbf{I}$, and $\mathbf{p}_c = \mathbf{0}$, $\mathbf{p}_\sigma = \mathbf{0}$,
 set $c_c \approx 4/n$, $c_\sigma \approx 4/n$, $c_{cov} \approx \mu_{eff}/n^2$, $\mu_{cov} = \mu_{eff}$, $d_\sigma \approx 1 + \sqrt{\frac{\mu_{eff}}{n}}$,
 λ , and $w_i, i = 1, \dots, \mu$ such that $\mu_{eff} \approx 0.3 \lambda$, where $\mu_{eff} = \frac{1}{\sum_{i=1}^{\mu} w_i^2}$

While not terminate

$$\mathbf{x}_i = \mathbf{m} + \sigma \mathbf{z}_i, \quad \mathbf{z}_i \sim \mathcal{N}_i(\mathbf{0}, \mathbf{C}), \quad \text{sampling}$$

$$\mathbf{m} \leftarrow \mathbf{m} + \sigma \langle \mathbf{z} \rangle_{\text{sel}} \quad \text{where } \langle \mathbf{z} \rangle_{\text{sel}} = \sum_{i=1}^{\mu} w_i \mathbf{z}_{i:\lambda} \quad \text{update mean}$$

$$\mathbf{p}_c \leftarrow (1 - c_c) \mathbf{p}_c + \mathbb{1}_{\{\|\mathbf{p}_\sigma\| < 1.5\sqrt{n}\}} \sqrt{1 - (1 - c_c)^2} \sqrt{\mu_{eff}} \langle \mathbf{z} \rangle_{\text{sel}} \quad \text{cumulation for } \mathbf{C}$$

$$\mathbf{C} \leftarrow (1 - c_{cov}) \mathbf{C} + c_{cov} \frac{1}{\mu_{cov}} \mathbf{p}_c \mathbf{p}_c^T \quad \text{update } \mathbf{C}$$

$$+ c_{cov} \left(1 - \frac{1}{\mu_{cov}}\right) \mathbf{Z} \quad \text{where } \mathbf{Z} = \sum_{i=1}^{\mu} w_i \mathbf{z}_{i:\lambda} \lambda \mathbf{z}_{i:\lambda}^T$$

$$\mathbf{p}_\sigma \leftarrow (1 - c_\sigma) \mathbf{p}_\sigma + \sqrt{1 - (1 - c_\sigma)^2} \sqrt{\mu_{eff}} \mathbf{C}^{-\frac{1}{2}} \langle \mathbf{z} \rangle_{\text{sel}} \quad \text{cumulation for } \sigma$$

$$\sigma \leftarrow \sigma \times \exp\left(\frac{c_\sigma}{d_\sigma} \left(\frac{\|\mathbf{p}_\sigma\|}{\mathbb{E}\|\mathcal{N}(\mathbf{0}, \mathbf{I})\|} - 1\right)\right) \quad \text{update of } \sigma$$

Survivors selection

Variants:

(μ, λ)

From the set of μ parents construct $\lambda > \mu$ offspring and starting from these select the best μ survivors (the number of offspring should be larger than the number of parents)

$(\mu + \lambda)$

From the set of μ parents construct λ offspring and from the joined population of parents and offspring select the best μ survivors (**truncation selection**). This is an **elitist** selection (it preserves the best element in the population)

Remark: if the number of parents is ρ the usual notations are:

$(\mu / \rho + \lambda)$

$(\mu / \rho, \lambda)$

Survivors selection

Particular cases:

$(1+1)$ – from one parent generate one offspring and chose the best one

$(1,+λ)$ – from one parent generate several offspring and choose the best element

$(μ+1)$ – from a set of $μ$ construct an offspring and insert it into population if it is better than the worst element in the population

Survivors selection

The variant ($\mu+1$) corresponds to the so called steady state (asynchronous) strategy

Generational strategy:

- At each generation is constructed a new population of offspring
- The selection is applied to the offspring or to the joined population
- This is a **synchronous process**

Steady state strategy:

- At each iteration only one offspring is generated; it is assimilated into population if it is good enough
- This is an **asynchronous process**

ES variants

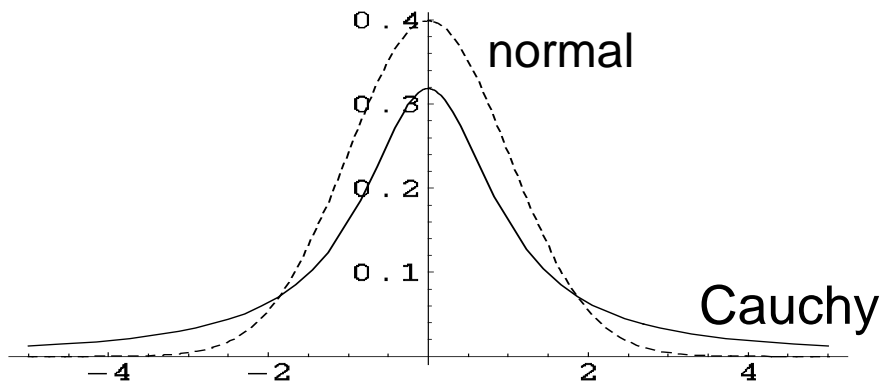
(μ, k, λ, ρ) strategies

Each element has a limited life time (k generations)

The recombination is based on λ parents

Fast evolution strategies:

The perturbation is based on the Cauchy distribution



$$\varphi(x) = \frac{s}{\pi(x^2 + s^2)}$$

Analysis of the behavior of ES

Evaluation criteria:

Effectiveness:

- Value of the objective function after a given number of evaluations (nfe)

Success ratio:

- The number of runs in which the algorithm reaches the goal divided by the total number of runs.

Efficiency:

- The number of evaluation functions necessary such that the objective function reaches a given value (a desired accuracy)

Summary

Encoding	Real vectors
Recombination	Discrete or intermediate
Mutation	Random additive perturbation (uniform, Gaussian, Cauchy)
Parents selection	Uniformly random
Survivors selection	(μ, λ) or $(\mu + \lambda)$
Particularity	Self-adaptive mutation parameters