Lab 6. MPI – solving linear systems and performance studies

Main goals of the assignment

- Ability to design a paralllel algorithm.
- Learn about performance through experiments.

The problem to solve

General overview

A MPI-based parallel version of Gauss-Jordan method to solve linear systems. To investigate its efficiency it should be compared with the sequential version of Gauss method.

Background

The main sequential method to obtain the exact solution of a linear system wth dense matrice is the Gauss's elimination method. This method has two stages: (1) reduce the system to a triangular shape; (2) regressive substitution.
An example for the system

$$\begin{cases} 2x_1 & +2x_2 & +4x_3 & +2x_4 & = 76 \\ x_1 & +3x_2 & +2x_3 & +x_4 & = 52 \\ 3x_1 & +2x_2 & +2x_3 & +x_4 & = 57 \\ x_1 & +x_2 & +3x_3 & +3x_4 & = 64 \end{cases}$$

Stage 1, step 1 – zeros on the first column: divide the first equation with the coefficient of x_1, multiply by the coedficient of x_1 from the second equation and substract it from the second equation; then proceed in a similar matter for the third and fourth equations. The result is the following – zeros on the first column, bellow the main diagonal:

$$\begin{cases} x_1 & +x_2 & +2x_3 & +x_4 & = 38 \\ & +2x_2 & +0x_3 & +0x_4 & = 14 \\ & -x_2 & -4x_3 & -2x_4 & = -57 \\ & +0x_2 & +x_3 & +2x_4 & = 26 \end{cases}$$

In step 2, zeros are produced on the second column:

$$\begin{cases} x_1 & +x_2 & +2x_3 & +x_4 & = 38 \\ & +x_2 & +0x_3 & +0x_4 & = 7 \\ & & -4x_3 & -2x_4 & = -50 \\ & & +x_3 & +2x_4 & = 26 \end{cases}$$

In step 3, zeros are produced on the third column:

$$\begin{cases} x_1 & +x_2 & +2x_3 & +x_4 & = 38 \\ & +x_2 & +0x_3 & +0x_4 & = 7 \\ & & x_3 & +0.5x_4 & = 12.5 \\ & & & +1.5x_4 & = 13.5 \end{cases}$$

In stage 2,

$$x_4 = 9, x_3 = 12.5 - 0.5x_4 = 8, x_2 = 7 - 0x_3 - 0x_4 = 7, x_1 = 38 - x_2 - 2x_3 - x_4 = 6$$

In general, for a dimension n of the system the algorithm of the Gauss method has a complexity of order $n^3/3$.
An imbalance is produced in the case when we assign the lines or bands of lines to different processing elements. Moreover, the stage 2 is difficult to be parallelized.

The method Gauss-Jordan that has a complexity of order $n^3/2$ is more appropriate for a parallel implementation. The main idea is make zeros also above the main diagonal (stage 2 is not necessary). In the case of the above mentioned system, the steps are producing the following intermediate systems:

$$\text{Step 1: } \left\{ \begin{array}{lllll} x_1 & +x_2 & +2x_3 & +x_4 & = 38 \\ & +2x_2 & +0x_3 & +0x_4 & = 14 \\ & -x_2 & -4x_3 & -2x_4 & = -57 \\ & +0x_2 & +x_3 & +2x_4 & = 26 \end{array} \right. \quad \text{Step 2: } \left\{ \begin{array}{lllll} x_1 & & +2x_3 & +x_4 & = 31 \\ & +x_2 & +0x_3 & +0x_4 & = 7 \\ & & -4x_3 & -2x_4 & = -50 \\ & & +x_3 & +2x_4 & = 26 \end{array} \right.$$

$$\text{Step 3: } \left\{ \begin{array}{lllll} x_1 & & & +0x_4 & = 6 \\ & +x_2 & & +0x_4 & = 7 \\ & & +x_3 & +0.5x_4 & = 12.5 \\ & & & +1.5x_4 & = 13.5 \end{array} \right. \quad \text{Step 4: } \left\{ \begin{array}{lllll} x_1 & & & & = 38 \\ & +x_2 & & & = 7 \\ & & +x_3 & & = 8 \\ & & & +x_4 & = 9 \end{array} \right.$$

Lines of bands of lines can be assigned to different processing elements, and the number of parallel steps is less than in the case of the parallel version of Gauss method.

To do

1. Write the code that implements the parallel version of the Gauss-Jordan method for solving the bellow system of equations for the dimension $n+1$ of hundreds' order.

   The system of equations that is proposed to be solved is $Vx = b$, where

$$V = \begin{pmatrix} a_0^{0/n} & a_1^{0/n} & \cdots & a_n^{0/n} \\ a_0^{1/n} & a_1^{1/n} & \cdots & a_n^{1/n} \\ a_0^{2/n} & a_1^{2/n} & \cdots & a_n^{2/n} \\ \vdots & \vdots & & \vdots \\ a_0^{n/n} & a_1^{n/n} & \cdots & a_n^{n/n} \end{pmatrix}, b = \begin{pmatrix} a_{n+1}^{0/n} \\ a_{n+1}^{1/n} \\ a_{n+1}^{2/n} \\ \vdots \\ a_{n+1}^{n/n} \end{pmatrix}$$

   where $a_0, a_1, \ldots, a_{n+1} \in (0,1)$ are distinct numbers.

2. Study the speedup (hint: compare with the sequential Gauss method for solving the same problem).